

# Admissible unitary completions of locally $\mathbb{Q}_p$ -rational representations of $\mathrm{GL}_2(F)$

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## Abstract

Let  $F$  be a finite extension of  $\mathbb{Q}_p$ ,  $p > 2$ . We construct admissible unitary completions of certain representations of  $\mathrm{GL}_2(F)$  on  $L$ -vector spaces, where  $L$  is a finite extension of  $F$ . When  $F = \mathbb{Q}_p$  using the results of Berger, Breuil and Colmez we obtain some results about lifting 2-dimensional mod  $p$  representations of the absolute Galois group of  $\mathbb{Q}_p$  to crystabelline representations with given Hodge-Tate weights.

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# 1 Introduction

Let  $F$  be a finite extension of  $\mathbb{Q}_p$  with the ring of integers  $\mathfrak{o}$ , uniformizer  $\varpi$  and the residue field isomorphic to  $\mathbb{F}_q$ . Let  $G := \mathrm{GL}_2(F)$  and  $K := \mathrm{GL}_2(\mathfrak{o})$ . Let  $L$  be a 'large' finite extension of  $\mathbb{Q}_p$ , ring of integers  $A$ ,  $\mathfrak{M}$  the maximal ideal in  $A$  and residue field  $k = k_L$ . Let  $\mathfrak{R}$  be the category of smooth representations of  $G$  on  $\overline{L}$ -vector spaces, then  $\mathfrak{R}$  decomposes into a product of subcategories  $\mathfrak{R} \cong \prod_{s \in \mathfrak{B}} \mathfrak{R}_s$ , where  $\mathfrak{B}$  is the set of inertial equivalence classes of supercuspidal representations of the Levy subgroups of  $G$ , see [6], [15]. Following Henniart [25, Def A.1.4.1] we say that an irreducible smooth  $\overline{L}$ -representation  $\tau$  of  $K$  is *typical* for the Bernstein component  $\mathfrak{R}_s$ , if for every irreducible object  $\pi$  in  $\mathfrak{R}$ ,  $\mathrm{Hom}_K(\tau, \pi) \neq 0$  implies that  $\pi$  lies in  $\mathfrak{R}_s$ . We say that  $\tau$  is a *type* for  $\mathfrak{R}_s$  if it is typical and  $\mathrm{Hom}_K(\tau, \pi) \neq 0$  for every irreducible object  $\pi$  in  $\mathfrak{R}_s$ . Given  $\mathfrak{R}_s$ , there exists a type  $\tau$ , unique up to isomorphism, except when  $\mathfrak{R}_s$  contains  $\chi \circ \det$ . In this case, there are two typical representations  $\theta \circ \det$  and  $\mathrm{St} \otimes \theta \circ \det$ , where  $\theta := \chi|_{\mathfrak{o}^\times}$  and  $\mathrm{St}$  is the lift to  $K$  of the Steinberg representations of  $\mathrm{GL}_2(\mathbb{F}_q)$ , see [25]. For us a  $\mathbb{Q}_p$ -rational representation of  $G$ , is a representation  $W$  of the form

$$\bigotimes_{\sigma: F \hookrightarrow L} (\mathrm{Sym}^{r_\sigma} L^2 \otimes \det^{a_\sigma})^\sigma, \quad (1)$$

where  $r_\sigma, a_\sigma$  are integers,  $r_\sigma \geq 0$ , and an element  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  in  $G$  acts on the  $\sigma$ -component via  $\begin{pmatrix} \sigma(a) & \sigma(b) \\ \sigma(c) & \sigma(d) \end{pmatrix}$ , see [11, §2] for a proper setting. The locally  $\mathbb{Q}_p$ -rational representations in the title refer to the representations of the form  $\pi \otimes_L W$ , where  $\pi$  is a smooth representation of  $G$  on an  $L$ -vector space and  $W$  is a  $\mathbb{Q}_p$ -rational representation as above.

**Theorem 1.1.** *Assume  $p > 2$ . Let  $\tau$  be a smooth absolutely irreducible  $L$ -representation of  $K$ , which is typical for the Bernstein component  $\mathfrak{R}_s$ . Let  $W$  be  $\mathbb{Q}_p$ -rational representation of  $G$ , twisted by a continuous character. Let  $M$  be a  $K$ -invariant lattice in  $\tau \otimes W$ . Let  $\kappa$  be an absolutely irreducible smooth admissible  $k$ -representation of  $G$ , such that  $\varpi$  acts trivially. Suppose that there exists an irreducible smooth  $k$ -representation  $\sigma$  of  $K$ , such that*

- (1)  $\mathrm{Hom}_K(\sigma, \kappa) \neq 0$ ;
- (2)  $\sigma$  occurs as a subquotient of  $M \otimes_A k$ .

Then there exists a finite extension  $L'$  of  $L$ , an absolutely irreducible smooth  $L'$ -representation  $\pi$  of  $G$  in  $\mathfrak{R}_s$ , and an admissible unitary  $L'$ -Banach space representation  $(E, \|\cdot\|)$  of  $G$ , such that the following hold:

- (i)  $\pi \otimes_{L'} W_{L'}$  is a dense  $G$ -invariant subspace of  $E$ ;
- (ii)  $\mathrm{Hom}_G(\kappa \otimes_k k', E^0 \otimes_{A'} k') \neq 0$ ,

where  $E^0$  is the unit ball in  $E$  with respect to  $\|\cdot\|$ .

We also have a variant of Theorem 1.1, when  $\tau$  is the trivial representation of  $K$ , which allows  $\pi$  to be possibly reducible unramified principal series representation, see Corollary 7.6. We do not know in general whether these completions are of finite length, and we can not control  $\pi$ , except that we know that  $\pi$  lies in  $\mathfrak{R}_s$ . However, we show that any admissible unitary completion arises from our construction, see Proposition 6.2 and Lemma 7.8. We also show that given  $\kappa$  as above, there exists a unitary admissible topologically irreducible  $L$ -Banach space representation  $E$  of  $G$ , such that  $\mathrm{Hom}_G(\kappa, E^0 \otimes_A k) \neq 0$ , where  $E^0$  is a unit ball in  $E$  with respect to a  $G$ -invariant norm defining the topology on  $E$ , see Corollary 6.3. This result means that if one decides to throw away some irreducible smooth  $k_L$ -representations of  $G$  by declaring them 'non-arithmetic', one is also forced to throw away some irreducible admissible unitary  $L$ -Banach space representations of  $G$ . When  $\mathfrak{R}_s$  contains a principal series representation, we show that in most cases the completions we get are not 'ordinary', for example when  $\kappa$  is supersingular. Topologically irreducible completions of locally  $\mathbb{Q}_p$ -rational representations are expected to be related to the 2-dimensional representations of the absolute Galois group of  $F$ , see [11]. If  $F = \mathbb{Q}_p$  this is indeed the case, see for example [4], [16], [17]. If  $F \neq \mathbb{Q}_p$  then there is not so much known about the completions of locally  $\mathbb{Q}_p$ -rational representations, with the exception of Vignéras paper [41]. However, the  $G$ -invariant lattices in  $\pi \otimes W$ , that one gets in [41] are always finitely generated over  $A[G]$ , it is expected that the completion with respect to such lattices will not be admissible in general.

If  $F = \mathbb{Q}_p$  and  $\mathfrak{R}_s$  contains a principal series representation then the results of Berger-Breuil [4] imply that the completions we get are topologically irreducible. Moreover, using results of Berger, Breuil and Colmez we may then transfer the statement of Theorem 1.1 to the Galois side. We will describe this in more detail. Recall that a representation  $V$  of  $\mathcal{G}_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$  is crystabelline if it becomes crystalline after restriction to  $\mathrm{Gal}(\overline{\mathbb{Q}_p}/E)$ , where  $E$  is an abelian extension of  $\mathbb{Q}_p$ . Absolutely irreducible

$L$ -linear 2-dimensional crystabelline representations of  $\mathcal{G}_{\mathbb{Q}_p}$  with Hodge-Tate weights  $(0, k-1)$ , ( $k \geq 2$ ) can be parameterized by pairs of smooth characters  $\alpha, \beta : \mathbb{Q}_p^\times \rightarrow L^\times$ , such that  $-(k-1) < \text{val}(\alpha(p)) \leq \text{val}(\beta(p)) < 0$  and  $\text{val}(\alpha(p)) + \text{val}(\beta(p)) = -(k-1)$ , see [4, Prop 2.4.5] or [16, §5.5]. We denote by  $V(\alpha, \beta)$  the unique crystabelline representation  $V$ , such that  $D_{\text{cris}}(V) = D(\alpha, \beta)$ , where  $D(\alpha, \beta)$  is the filtered admissible  $L$ -linear  $(\varphi, \mathcal{G}_{\mathbb{Q}_p})$ -module defined in [4, Def 2.4.4]. We denote by  $\overline{V}$  the semi-simplification of the reduction modulo  $\mathfrak{M}$  of any  $\mathcal{G}_{\mathbb{Q}_p}$ -stable lattice in  $V$ , let  $\mathcal{I}_{\mathbb{Q}_p}$  be the inertia subgroup of  $\mathcal{G}_{\mathbb{Q}_p}$  and  $\omega$  the reduction modulo  $p$  of the cyclotomic character.

**Theorem 1.2.** *Assume  $p > 2$ . Fix smooth characters  $\theta_1, \theta_2 : \mathbb{Z}_p^\times \rightarrow L^\times$ , and an integer  $k \geq 2$ , such that*

- (a) *if  $\theta_1 = \theta_2$  then assume  $k \geq p^2 + 1$ ;*
- (b) *if  $\theta_1 \neq \theta_2$  and  $\theta_1 \theta_2^{-1}$  is trivial on  $1 + p\mathbb{Z}_p$  then assume  $k \geq p$ .*

*Let  $\rho$  be a semisimple 2-dimensional  $k_L$ -representation of  $\mathcal{G}_{\mathbb{Q}_p}$ , such that*

- (c)  *$\det \rho|_{\mathcal{I}_{\mathbb{Q}_p}} = \bar{\theta}_1 \bar{\theta}_2 \omega^{k-1}$ ;*
- (d) *if  $\rho$  is irreducible, then it is absolutely irreducible;*
- (e)  *$\rho|_{\mathcal{I}_{\mathbb{Q}_p}} \not\cong \bar{\theta}_1 \oplus \bar{\theta}_2 \omega^{k-1}$  and  $\rho|_{\mathcal{I}_{\mathbb{Q}_p}} \not\cong \bar{\theta}_2 \oplus \bar{\theta}_1 \omega^{k-1}$ .*

*Then there exists a finite extension  $L'$  of  $L$  and an absolutely irreducible 2-dimensional crystabelline  $L'$ -representation  $V := V(\alpha, \beta)$  of  $\mathcal{G}_{\mathbb{Q}_p}$ , such that*

- (i)  *$\overline{V} \cong \rho$ ;*
- (ii) *Hodge-Tate weights of  $V$  are  $(0, k-1)$ ;*
- (iii) *either  $(\alpha|_{\mathbb{Z}_p^\times} = \theta_1 \text{ and } \beta|_{\mathbb{Z}_p^\times} = \theta_2)$  or  $(\alpha|_{\mathbb{Z}_p^\times} = \theta_2 \text{ and } \beta|_{\mathbb{Z}_p^\times} = \theta_1)$ .*

See Theorem 8.6 for all  $k \geq 2$ . In the Example in §8 we check that our theory matches the known reductions of crystalline representations of small weights. To get the result when  $\theta_1 = \theta_2$  we need to get around the case of equal Frobenius eigenvalues, which is not treated in the literature, this is done in [32]. We comment on the assumptions on  $\rho$  in Theorem 1.2: (c) is necessary, (d) is not serious, since if  $\rho$  is irreducible then it is either absolutely irreducible or becomes reducible semi-simple after replacing  $k_L$  with a finite extension, we impose (e) to make sure that we stay out of the 'ordinary' case, i.e. to ensure that the representation  $V$  we get is absolutely irreducible.

Since we cannot control  $\pi$  in Theorem 1.1, we cannot control  $\alpha(p)$  and  $\beta(p)$  in Theorem 1.2. The condition  $\pi$  lies in  $\mathfrak{R}_s$  in Theorem 1.1 translates into condition (iii) in Theorem 1.2.

We will sketch the proof of Theorem 1.1. Let  $e$  be an edge on the Bruhat-Tits tree, containing a vertex  $v$ . Let  $\mathfrak{K}_1$  be the  $G$ -stabilizer of  $e$  and  $\mathfrak{K}_0$  the  $G$ -stabilizer of  $v$ . The key point in our construction is that  $G$  is an amalgam of  $\mathfrak{K}_0$  and  $\mathfrak{K}_1$  along  $\mathfrak{K}_0 \cap \mathfrak{K}_1$ , which is the stabilizer of  $e$  preserving the orientation. This is used in [30] and [12] to construct irreducible  $k_L$ -representations. We may assume that  $\mathfrak{K}_0 = KZ$ , where  $Z \cong F^\times$  is the centre of  $G$ . Let  $\kappa$  be as in Theorem 1.1, then in [12] it is shown that there exists a  $G$ -equivariant injection  $\kappa \hookrightarrow \Omega$ , such that  $\varpi \in Z$  acts trivially on  $\Omega$ ,  $\Omega|_K$  is an injective envelope of  $\kappa$  in the category  $\text{Rep}_{k_L} K$  of smooth  $k_L$ -representation of  $K$ . (For injective/projective envelopes see §3.) Since  $\kappa$  is admissible, so is  $\Omega$ , moreover  $\text{soc}_G \Omega \cong \kappa$ . Recall that the socle  $\text{soc}$  is the maximal semi-simple subobject. The first step is to lift  $\Omega$  to a unitary admissible Banach space representation, §6.

We start with the general discussion, the details are contained in §4, §5. Let  $\mathcal{G}$  be a compact  $p$ -adic analytic group and let  $I$  be an admissible injective object in  $\text{Rep}_{k_L} \mathcal{G}$ . Then dually  $I^\vee$  is a projective finitely generated module of the completed group algebra  $k[[\mathcal{G}]]$ , we may lift  $I^\vee$  to a projective finitely generated module  $P$  of  $A[[\mathcal{G}]]$ . This module  $P$  is then unique up to isomorphism. To  $P$  following Schneider-Teitelbaum [36] we may associate a unitary admissible  $L$ -Banach space representation  $P^d$  of  $\mathcal{G}$ ,  $P^d := \text{Hom}_A^{\text{cont}}(P, L)$ , with the supremum norm. If we let  $(P^d)^0$  be the unit ball in  $P^d$ , then  $(P^d)^0 \otimes_A k_L \cong I$  as  $G$ -representations. Concretely, when  $\mathcal{G}$  is a pro- $p$  group, then the only irreducible smooth  $k_L$ -representation is the trivial one, and so  $I$  is a finite direct sum of injective envelopes of the trivial representation. If  $I$  is an injective envelope of the trivial representation then  $I \cong C(\mathcal{G}, k_L)$ , the space of continuous function from  $\mathcal{G}$  to  $k_L$ ,  $I^\vee \cong k_L[[\mathcal{G}]]$ ,  $P \cong A[[\mathcal{G}]]$  and  $P^d \cong C(\mathcal{G}, L)$ , the space of continuous functions from  $\mathcal{G}$  to  $L$  with the supremum norm.

We now go back to  $\Omega$ . For  $i \in \{0, 1\}$ , set  $\mathcal{G}_i := \mathfrak{K}_i / \varpi^\mathbb{Z}$ , since  $\varpi$  acts trivially,  $\Omega$  is a representation of  $\mathcal{G}_i$ . Denote the restriction of  $\Omega$  to  $\mathcal{G}_i$  by  $\Omega_i$ . The assumption  $p \neq 2$  implies that the pro- $p$  Sylow subgroup of  $\mathcal{G}_1$  is equal to the pro- $p$  Sylow subgroup of  $\mathcal{G}_1 \cap \mathcal{G}_0$ , which is a pro- $p$  Sylow subgroup of  $\mathcal{G}_0$ . This implies that  $\Omega_1$  is an admissible injective object in  $\text{Rep}_{k_L} \mathcal{G}_1$ . The argument above gives us a projective finitely generated module  $P_i$  of  $A[[\mathcal{G}_i]]$ . Using some general facts about projective modules we find a  $(\mathcal{G}_0 \cap \mathcal{G}_1)$ -equivariant isomorphism  $\phi : P_0 \cong P_1$ , such that  $\phi$  reduces to the identity modulo  $\mathfrak{M}$ . The results of [36] enable us to go back and forth between finitely generated  $A[[\mathcal{G}_i]]$ -modules, and admissible unitary  $L$ -Banach space representations. So

dually we get unitary  $L$ -Banach space representation  $P_0^d$  of  $\mathcal{G}_0$ ,  $P_1^d$  of  $\mathcal{G}_1$ , and a  $(\mathcal{G}_1 \cap \mathcal{G}_0)$ -equivariant isometrical isomorphism  $\phi^d : P_1^d \xrightarrow{\cong} P_0^d$ . We let  $\varpi$  act trivially everywhere, then by the amalgamation argument this data glues to a unitary admissible  $L$ -Banach space representation  $B$  of  $G$ . Moreover,  $B^0 \otimes_A k_L \cong \Omega$  as a  $G$ -representation. We note that although  $P_0$  and  $P_1$  are canonical, there is no canonical way to choose the isomorphism  $\phi$ . In general, different choices of  $\phi$  will lead to non-isomorphic Banach space representations  $B$ , and different  $\pi$  in Theorem 1.1.

The second step is to produce  $\pi$ , see §7. The assumption  $\text{Hom}_K(\sigma, \kappa) \neq 0$  implies that an injective envelope  $I_\sigma$  of  $\sigma$  is a direct summand of  $\Omega|_K$ . This implies that the projective envelope  $P_{\sigma^*}$  of  $\sigma^*$  is a direct summand of  $P_0$ . We show that the assumption (2) in Theorem 1.1 implies that

$$\text{Hom}_{L[[\mathcal{G}_0]]}(P_0 \otimes_A L, (\tau \otimes W)^*) \neq 0,$$

where  $L[[\mathcal{G}_0]] := L \otimes_A A[[\mathcal{G}_0]]$ . Dually this means  $\text{Hom}_K(\tau \otimes W, B) \neq 0$ . Since  $\varpi$  acts trivially on  $B$  and by a scalar on  $W$ , there exists a unique extension  $\tilde{\tau}$  of  $\tau$  to a representation of  $\mathfrak{K}_0$ , such that  $\text{Hom}_{\mathfrak{K}_0}(\tilde{\tau} \otimes W, B) \neq 0$ . Frobenius reciprocity then gives

$$\text{Hom}_G(\text{c-Ind}_{\mathfrak{K}_0}^G \tilde{\tau} \otimes W, B) \cong \text{Hom}_G((\text{c-Ind}_{\mathfrak{K}_0}^G \tilde{\tau}) \otimes W, B) \neq 0. \quad (2)$$

Using admissibility of  $B$ , we show that  $B$  contains a  $G$ -invariant subspace of the form  $\pi' \otimes W$ , where  $\pi'$  is a quotient of  $\text{c-Ind}_{\mathfrak{K}_0}^G \tilde{\tau}$  of finite length. Thus if we replace  $L$  with a finite extension, we may find a  $G$ -invariant subspace in  $B$  isomorphic to  $\pi \otimes W$ , where  $\pi$  is an absolutely irreducible smooth representation of  $G$ . Since  $\tau$  is typical for  $\mathfrak{R}_s$ ,  $\pi_{\overline{L}}$  is an object of  $\mathfrak{R}_s$ . Since we work with coefficient fields which are not algebraically closed we use the results of Vignéras [39]. Take  $E$  to be the closure of  $\pi \otimes W$  in  $B$ , then since  $B$  is admissible, so is  $E$  and we have an injection  $E^0 \otimes_A k_L \hookrightarrow B^0 \otimes_A k \cong \Omega$ . Since by construction  $\text{soc}_G \Omega \cong \kappa$ , this yields the result.

In general, it is quite hard to compute inside  $B$ . However, it might be possible to understand the completions better if we restrict ourselves to the case when  $\tau$  is the trivial representation, so that  $\pi$  is an unramified principal series, and the weights in (1) are small,  $0 \leq r_\sigma \leq p-1$ . Using our methods one could try and lift the representations constructed in [12] to Banach space representations, (at least those that conjecturally correspond to the irreducible mod  $p$  representations of  $\text{Gal}(\overline{F}/F)$ ), see Remark 7.7. We hope to return to these questions in the future work.

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## 2 Notation

Let  $F$  be a finite extension of  $\mathbb{Q}_p$  with the ring of integers  $\mathfrak{o}$ , maximal ideal  $\mathfrak{p}$  and the residue field isomorphic to  $\mathbb{F}_q$ . We fix a uniformizer  $\varpi$  of  $F$ . Let  $G := \mathrm{GL}_2(F)$ ,  $B$  the subgroup of upper-triangular matrices,  $U$  the subgroup of unipotent upper-triangular matrices,  $K := \mathrm{GL}_2(\mathfrak{o})$ ,

$$I := \begin{pmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o}^\times \end{pmatrix}, \quad I_1 := \begin{pmatrix} 1 + \mathfrak{p} & \mathfrak{o} \\ \mathfrak{p} & 1 + \mathfrak{p} \end{pmatrix}, \quad K_1 := \begin{pmatrix} 1 + \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & 1 + \mathfrak{p} \end{pmatrix}.$$

Let

$$s := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Pi := \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}, \quad t := \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}.$$

Let  $Z$  be the centre of  $G$ ,  $Z \cong F^\times$ . Let  $\mathfrak{K}_0$  be the  $G$ -normalizer of  $K$ , so that  $\mathfrak{K}_0 = KZ$  and let  $\mathfrak{K}_1$  be the  $G$ -normalizer of  $I$ , so that  $\Pi$  and  $I$  generate  $\mathfrak{K}_1$  as a group.

We fix an algebraic closure  $\overline{\mathbb{Q}_p}$  of  $\mathbb{Q}_p$ . We let  $\mathrm{val}$  be the valuation on  $\overline{\mathbb{Q}_p}$  such that  $\mathrm{val}(p) = 1$ , and we set  $|x| := p^{-\mathrm{val}(x)}$ . Let  $L$  be a finite extension of  $\mathbb{Q}_p$  contained in  $\overline{\mathbb{Q}_p}$ ,  $A$  the ring of integers of  $L$ ,  $\varpi_L$  a uniformizer, and  $\mathfrak{M}$  the maximal ideal of  $A$ ,  $k = k_L$  the residue field. The field  $L$  will be our coefficient field, when needed we replace  $L$  by a finite extension. Let  $\Sigma$  be the set of  $\mathbb{Q}_p$ -linear embeddings  $F \hookrightarrow L$ , we assume  $[F : \mathbb{Q}_p] = |\Sigma|$ . Note that  $k_L$  contains  $\mathbb{F}_q$ . As a consequence every irreducible smooth  $k_L$ -representation of  $K$  is absolutely irreducible. If  $M$  is an  $A$ -module then we write  $\overline{M} := M \otimes_A k$ . If  $V$  is a vector space over some field  $F$  we write  $V^* := \mathrm{Hom}_F(V, F)$  and if  $F'$  is a field extension of  $F$ , then  $V_{F'} := V \otimes_F F'$ .

## 3 Injective and projective envelopes

We recall some standard facts about injective and projective envelopes. Let  $\mathcal{A}$  be an abelian category. A monomorphism  $\iota : N \hookrightarrow M$  is *essential* if for every

non-zero subobject  $M_1$  of  $M$  we have  $N \cap M_1 \neq 0$ . An injective envelope of  $M$  is an essential monomorphism  $\iota : M \hookrightarrow I$ , such that  $I$  is an injective object in  $\mathcal{A}$ . An epimorphism  $q : M \twoheadrightarrow N$  is *essential* if for every morphism  $s : P \rightarrow M$ , the assertion ' $qs$  is an epimorphism' implies that  $s$  is an epimorphism. A projective envelope of  $M$  is an essential epimorphism  $q : P \twoheadrightarrow M$  with  $P$  a projective object in  $\mathcal{A}$ . One may easily verify that injective and projective envelopes (if they exist) are unique up to (non-unique) isomorphism. So by abuse of language we will forget the morphism and say  $I$  is an injective envelope of  $M$  or  $P$  is a projective envelope of  $M$ .

**Theorem 3.1.** *Let  $R$  be a ring and  $\text{Mod}(R)$  the category of left  $R$ -modules, then every object in  $\text{Mod}(R)$  has an injective envelope.*

*Proof.* [28] Theorem 11.3. □

**Lemma 3.2.** *Let  $\mathcal{A}'$  be a full abelian subcategory of  $\mathcal{A}$  and assume that we have a functor  $F : \mathcal{A} \rightarrow \mathcal{A}'$ , which is right adjoint to the inclusion  $i : \mathcal{A}' \rightarrow \mathcal{A}$ . Let  $M$  be an object in  $\mathcal{A}'$  and suppose that  $M$  has an injective envelope  $i(M) \hookrightarrow I$  in  $\mathcal{A}$ , then  $M \hookrightarrow F(I)$  is an injective envelope of  $M$  in  $\mathcal{A}'$ .*

*Proof.* We have  $\text{Hom}_{\mathcal{A}'}(M, N) = \text{Hom}_{\mathcal{A}}(i(M), i(N)) = \text{Hom}_{\mathcal{A}'}(M, F(i(N)))$ . So  $F \circ i$  is right adjoint to the identity, and hence  $N$  is canonically isomorphic to  $F(i(N))$  for all  $N$  in  $\mathcal{A}'$ . Let  $\alpha : i(M) \rightarrow N$  be a morphism in  $\mathcal{A}$ . Then for all  $M'$  in  $\mathcal{A}'$  the map  $\alpha^* : \text{Hom}_{\mathcal{A}}(i(M'), i(M)) \rightarrow \text{Hom}_{\mathcal{A}}(i(M'), N)$  is an injection if and only if the map  $F(\alpha)^* : \text{Hom}_{\mathcal{A}'}(M', M) \rightarrow \text{Hom}_{\mathcal{A}'}(M', F(N))$  is an injection. So  $F$  maps monomorphism to monomorphism, and for all  $N$  in  $\mathcal{A}$  the canonical map  $\iota_F : i(F(N)) \rightarrow N$  is a monomorphism. Moreover, given  $\phi \in \text{Hom}_{\mathcal{A}}(i(M), N)$ , we have  $\phi = \iota_F \circ i(F(\phi))$ , as  $F(\iota_F \circ i(F(\phi))) = \text{id}_{\mathcal{A}'} \circ F(\phi) = F(\phi)$ . Let  $i(M) \hookrightarrow N$  be an essential monomorphism in  $\mathcal{A}$ , then it factors through  $i(M) \hookrightarrow i(F(N)) \hookrightarrow N$ , which implies that  $M \hookrightarrow F(N)$  is an essential monomorphism in  $\mathcal{A}'$ . Since  $I$  is injective the functor  $\text{Hom}_{\mathcal{A}}(\cdot, I)$  is exact, hence the functor  $\text{Hom}_{\mathcal{A}'}(\cdot, F(I))$  is exact so  $F(I)$  is injective in  $\mathcal{A}'$ . □

Using the theorem and the lemma one can obtain a lot of injective envelopes. We give some examples.

1) Let  $\mathcal{G}$  be a topological group. We say that a representation of  $\mathcal{G}$  on a  $k$ -vector space  $V$  is smooth (or discrete), if the action of  $\mathcal{G}$  on  $V$  is continuous, for the discrete topology on  $V$ . This is equivalent to saying that for all  $v \in V$  the stabilizer of  $v$  is an open subgroup of  $\mathcal{G}$ . We denote the category of smooth  $k$ -representations of  $\mathcal{G}$  by  $\text{Rep}_k(\mathcal{G})$ . We may view  $\text{Rep}_k(\mathcal{G})$  as



a full subcategory of  $\text{Mod}(k[\mathcal{G}])$ . If  $M$  is in  $\text{Mod}(k[\mathcal{G}])$  we let  $F(M)$  be a submodule of  $M$  consisting of  $v \in M$  such that the orbit map  $\mathcal{G} \rightarrow M$ ,  $g \mapsto gv$  is continuous, for the discrete topology on  $M$ . Then  $F$  satisfies the conditions of the Lemma 3.2, and hence every object in  $\text{Rep}_k(\mathcal{G})$  has an injective envelope.

2) Let  $\mathfrak{D}_A(\mathcal{G})$  be the category of  $p$ -torsion  $A$ -modules  $M$  with a continuous action of  $\mathcal{G}$ , for the discrete topology on  $M$ . Then  $\mathfrak{D}_A(\mathcal{G})$  is a full subcategory of  $\text{Mod}(A[\mathcal{G}])$ , and  $F(M)$  is a submodule of  $M$  consisting of  $v$ , for which the orbit map is continuous and which are killed by some power of  $p$ . Again we obtain that every object in  $\mathfrak{D}_A(\mathcal{G})$  has an injective envelope.

Projective envelopes are harder to come by, but in our situation we have two abelian categories  $\mathcal{A}$  and  $\mathcal{A}^\vee$  and a functor  $M \mapsto M^\vee$  which induces an anti-equivalence of categories between  $\mathcal{A}$  and  $\mathcal{A}^\vee$ . One may check that

**Lemma 3.3.** *A monomorphism  $M \hookrightarrow I$  in  $\mathcal{A}$  is an injective envelope of  $M$  in  $\mathcal{A}$  if and only if  $I^\vee \twoheadrightarrow M^\vee$  is a projective envelope of  $M^\vee$  in  $\mathcal{A}^\vee$ .*

In the following let  $\mathcal{G}$  be a profinite group with an open pro- $p$  subgroup. We discuss the structure of injective envelopes in  $\text{Rep}_k(\mathcal{G})$ .

**Definition 3.4.** *A smooth  $k$ -representation  $V$  of  $\mathcal{G}$  is called admissible if for every open pro- $p$  subgroup  $\mathcal{P}$  of  $\mathcal{G}$  the subspace  $V^\mathcal{P} := \{v \in V : gv = v, \forall g \in \mathcal{P}\}$  is finite dimensional.*

In fact, it is enough to check this for one open pro- $p$  group, see [30] Theorem 6.3.2. If  $\mathcal{P}$  is an open normal pro- $p$  group of  $\mathcal{G}$  and  $S$  is irreducible, then  $\mathcal{P}$  acts trivially on  $S$ , since  $S^\mathcal{P}$  is a non-zero subrepresentation of  $S$ . The irreducible representations of  $\mathcal{G}$  coincide with the irreducible representations of a finite group  $\mathcal{G}/\mathcal{P}$ . In particular, the set  $\text{Irr}(\mathcal{G})$  of the irreducible representations is finite.

**Lemma 3.5.** *Let  $V$  be a smooth representation of  $\mathcal{G}$  then  $V$  is admissible if and only if the space  $\text{Hom}_\mathcal{G}(S, V)$  is finite dimensional for all irreducible representations  $S$  of  $\mathcal{G}$ .*

*Proof.* Let  $\mathcal{P}$  be an open normal pro- $p$  subgroup of  $\mathcal{G}$ , then since  $\mathcal{P}$  acts trivially on  $S$ , we have  $\text{Hom}_\mathcal{G}(S, V) \cong \text{Hom}_\mathcal{G}(S, V^\mathcal{P})$  and this space is finite dimensional. Suppose that  $\text{Hom}_\mathcal{G}(S, V)$  is finite dimensional for all irreducible representations  $S$ . Then arguing inductively we get that  $\text{Hom}_\mathcal{G}(M, V)$  is finite dimensional for all representations  $M$  of finite length. In particular,

$$V^\mathcal{P} \cong \text{Hom}_\mathcal{P}(\mathbf{1}, V) \cong \text{Hom}_\mathcal{G}(\text{Ind}_\mathcal{P}^\mathcal{G} \mathbf{1}, V)$$

is finite dimensional. □

**Lemma 3.6.** *Let  $V$  be an admissible smooth representation of  $\mathcal{G}$ . For each irreducible representation  $S$  of  $\mathcal{G}$  set  $m_S := \dim_k \operatorname{Hom}_{\mathcal{G}}(S, V)$ . Let  $V \hookrightarrow I$  be an injective envelope of  $V$  in  $\operatorname{Rep}_k(\mathcal{G})$ . Then  $I$  is admissible and  $I \cong \bigoplus_{S \in \operatorname{Irr}(\mathcal{G})} I_S^{\oplus m_S}$ , where  $I_S$  is an injective envelope of  $S$  in  $\operatorname{Rep}_k(\mathcal{G})$ .*

*Proof.* For each irreducible  $S$  we have  $\operatorname{Hom}_{\mathcal{G}}(S, V) \cong \operatorname{Hom}_{\mathcal{G}}(S, I)$ , otherwise  $S$  would be a nonzero subspace of  $I$ , such that  $S \cap V = 0$ . Lemma 3.5 implies that  $I$  is admissible. So the maximal semisimple subobject  $\operatorname{soc}_{\mathcal{G}} I$  (socle) of  $I$ , is isomorphic to  $\bigoplus S^{\oplus m_S}$ . Now  $I$  is an essential extension of  $\operatorname{soc}_{\mathcal{G}} I$ , since if  $W$  is a  $\mathcal{G}$ -invariant subspace of  $I$  such that  $W \cap \operatorname{soc}_{\mathcal{G}} I = 0$  then  $\operatorname{soc}_{\mathcal{G}} W = 0$ . This implies that if  $\mathcal{P}$  is an open normal subgroup of  $\mathcal{G}$  then  $W^{\mathcal{P}} = 0$ , and hence  $W = 0$ . One easily checks that  $\bigoplus_{S \in \operatorname{Irr}(\mathcal{G})} I_S^{\oplus m_S}$  is an injective envelope of  $\bigoplus S^{\oplus m_S}$ , and the uniqueness of injective envelopes implies the claim.  $\square$

**Lemma 3.7.** *Let  $\mathcal{P}$  be a pro- $p$  group and let  $C(\mathcal{P}, k)$  be the space of continuous functions from  $\mathcal{P}$  to  $k$ . Then  $\mathbf{1} \hookrightarrow C(\mathcal{P}, k)$  is an injective envelope of  $\mathbf{1}$  in  $\operatorname{Rep}_k(\mathcal{G})$ .*

*Proof.* Now  $C(\mathcal{P}, k)^{\mathcal{P}}$  is just the space of constant functions, so it is one dimensional. Hence,  $\mathbf{1} \hookrightarrow C(\mathcal{P}, k)$  is an essential monomorphism. If  $V$  in  $\operatorname{Rep}_k(\mathcal{G})$  then  $\operatorname{Hom}_{\mathcal{G}}(V, C(\mathcal{P}, k)) \cong \operatorname{Hom}_k(V, k)$  by Frobenius reciprocity. Hence the functor  $\operatorname{Hom}_{\mathcal{G}}(\bullet, C(\mathcal{P}, k))$  is exact and so  $C(\mathcal{P}, k)$  is injective.  $\square$

**Lemma 3.8.** *Let  $\mathcal{G}$  be a profinite group with an open pro- $p$  subgroup. Suppose that every irreducible smooth  $k$ -representation of  $\mathcal{G}$  is absolutely irreducible. Let  $S$  be an irreducible smooth  $k$ -representation of  $\mathcal{G}$  and  $I_S$  be an injective envelope of  $S$  in  $\operatorname{Rep}_k \mathcal{G}$ . Let  $k'$  be an extension of  $k$ , then  $I_S \otimes_k k'$  is an injective envelope of  $S \otimes_k k'$  in  $\operatorname{Rep}_{k'} \mathcal{G}$ .*

*Proof.* Lemma 3.6 gives an isomorphism  $C(\mathcal{G}, k) \cong \bigoplus_S I_S^{\dim_k S}$ . Now

$$C(\mathcal{G}, k') \cong C(\mathcal{G}, k) \otimes_k k' \cong \bigoplus_S (I_S \otimes_k k')^{\dim_k S}.$$

Since  $I_S \otimes_k k'$  is a direct summand of an injective object in  $\operatorname{Rep}_{k'} \mathcal{G}$ , it is also injective. Since we have an injection  $S \otimes_k k' \hookrightarrow I_S \otimes_k k'$ , the injective envelope  $I_{S \otimes_k k'}$  of  $S \otimes_k k'$  will be isomorphic to a direct summand of  $I_S \otimes_k k'$ . By assumption  $S \otimes_k k'$  is irreducible, hence  $C(\mathcal{G}, k') \cong \bigoplus_S I_{S \otimes_k k'}^{\dim_k S}$ . This implies  $I_{S \otimes_k k'} \cong I_S \otimes_k k'$ .  $\square$

**Lemma 3.9.** *Let  $\mathcal{G}$  be a profinite group with an open pro- $p$  subgroup. Let  $V$  be an admissible  $k$ -representation of  $\mathcal{G}$ , and let  $k'$  be an extension of  $k$ . Suppose that every irreducible  $k$ -representation of  $\mathcal{G}$  is absolutely irreducible then  $(\operatorname{soc}_{\mathcal{G}} V) \otimes_k k' \cong \operatorname{soc}_{\mathcal{G}}(V \otimes_k k')$ .*

*Proof.* For an irreducible smooth  $k$ -representation  $S$  of  $\mathcal{G}$ , set

$$m_S := \dim_k \operatorname{Hom}_{\mathcal{G}}(S, V), \quad m'_S := \dim_{k'} \operatorname{Hom}_{\mathcal{G}}(S \otimes_k k', V \otimes_k k').$$

We have to show that  $m_S = m'_S$  for all  $S$ . Clearly  $m_S \leq m'_S$ . Let  $I$  be an injective envelope of  $V$  in  $\operatorname{Rep}_k \mathcal{G}$  and let  $I'$  be an injective envelope of  $V \otimes_k k'$  in  $\operatorname{Rep}_{k'} \mathcal{G}$ . It follows from Lemmas 3.6 and 3.8 that  $I \otimes_k k'$  is injective in  $\operatorname{Rep}_{k'} \mathcal{G}$ . Since we have an injection  $V \otimes_k k' \hookrightarrow I \otimes_k k'$ ,  $I'$  is isomorphic to a direct summand of  $I \otimes_k k'$ . Lemmas 3.6 and 3.8 imply that  $m'_S \leq m_S$ .  $\square$

**Lemma 3.10.** *Let  $\mathcal{P}$  be a pro- $p$  group and let  $V$  be a smooth admissible  $k$ -representation of  $\mathcal{P}$ . Let  $k'$  be an extension of  $k$ , then  $V^{\mathcal{P}} \otimes_k k' \cong (V \otimes_k k')^{\mathcal{P}}$ .*

*Proof.* The only irreducible smooth  $k$ -representation of  $\mathcal{P}$  is the trivial representation, which is absolutely irreducible. Since  $\operatorname{soc}_{\mathcal{P}} V = V^{\mathcal{P}}$  Lemma 3.9 implies the assertion.  $\square$

## 4 Modules over completed group algebras

Let  $\mathcal{G}$  be a pro-finite group with an open pro- $p$  subgroup. We define the completed group algebras:

$$A[[\mathcal{G}]] := \varprojlim_{\leftarrow} A[\mathcal{G}/\mathcal{P}] \cong \varprojlim_{\leftarrow} A/\mathfrak{M}^n[\mathcal{G}/\mathcal{P}], \quad k[[\mathcal{G}]] := \varprojlim_{\leftarrow} k[\mathcal{G}/\mathcal{P}],$$

where the limit runs over all open normal pro- $p$  subgroups and natural numbers  $n$ . We put the discrete topology on  $A/\mathfrak{M}^n[\mathcal{G}/\mathcal{P}]$  (resp.  $k[\mathcal{G}/\mathcal{P}]$ ) and inverse limit topology on  $A[[\mathcal{G}]]$  (resp.  $k[[\mathcal{G}]]$ ). So for all open normal pro- $p$  subgroups  $\mathcal{P}$  and all  $n \geq 1$  the kernels of  $A[[\mathcal{G}]] \rightarrow A/\mathfrak{M}^n[\mathcal{G}/\mathcal{P}]$  (resp.  $k[[\mathcal{G}]] \rightarrow k[\mathcal{G}/\mathcal{P}]$ ) form a basis of open neighbourhoods of zero. Since  $k$  is a finite field  $A/\mathfrak{M}^n[\mathcal{G}/\mathcal{P}]$  (resp.  $k[\mathcal{G}/\mathcal{P}]$ ) is finite, hence  $A[[\mathcal{G}]]$  (resp.  $k[[\mathcal{G}]]$ ) is compact. In the following let  $\Lambda$  be either  $A[[\mathcal{G}]]$  or  $k[[\mathcal{G}]]$ . Let  $\mathfrak{C}(\Lambda)$  denote the category of topological, Hausdorff, complete  $\Lambda$ -modules  $M$ , such that  $M$  has a system of open neighbourhoods of 0 consisting of submodules  $N$ , for which  $M/N$  has finite length, with morphisms continuous  $\Lambda$ -homomorphisms. Since  $k$  is a finite field, such modules  $M$  are compact. Recall that a topological  $\Lambda$ -module  $M$  is linearly topological if 0 has a fundamental system of open neighbourhoods consisting of  $\Lambda$ -submodules. Equivalently the category  $\mathfrak{C}(\Lambda)$  could be defined as a category of linearly topological compact Hausdorff  $\Lambda$ -modules. Let  $\mathfrak{D}(\Lambda)$  be the category of discrete  $\Lambda$ -modules, a discrete module is always  $p$ -torsion. Moreover,  $\mathfrak{D}(k[[\mathcal{G}]]) = \operatorname{Rep}_k(\mathcal{G})$ .

We recall some facts about these categories. The category  $\mathfrak{C}(\Lambda)$  is abelian with exact inverse limits, [24, Thm 3]. The category  $\mathfrak{D}(\Lambda)$  is abelian with exact direct limits, [13] Lem 1.8. If  $M$  is an object of  $\mathfrak{D}(\Lambda)$  or  $\mathfrak{C}(\Lambda)$ , then define  $M^\vee := \text{Hom}_A^{\text{cont}}(M, L/A)$ , with the discrete topology on  $L/A$  and compact open topology on  $M^\vee$ . Then  $M^{\vee\vee} \cong M$  and the functor  $\text{Hom}_A^{\text{cont}}(\cdot, L/A)$  induces an anti-equivalence of categories between  $\mathfrak{C}(\Lambda)$  and  $\mathfrak{D}(\Lambda)$ .

Every object in  $\mathfrak{D}(\Lambda)$  has an injective envelope, dually every object in  $\mathfrak{C}(\Lambda)$  has a projective envelope. Let  $M$  and  $N$  be projective objects in  $\mathfrak{C}(\Lambda)$ . Denote by  $\text{rad}(M)$  the intersection of all maximal subobjects in  $M$ . Then  $M \cong N$  if and only if  $M/\text{rad}(M) \cong N/\text{rad}(N)$ . Moreover, a projective object is indecomposable if and only if  $M/\text{rad}(M)$  is simple, [19] V.2.4.6 b). This is equivalent to: a projective module is indecomposable if and only if it is a projective envelope of an irreducible module. Every indecomposable projective module in  $\mathfrak{C}(\Lambda)$  is isomorphic to a direct summand of  $\Lambda$ , [24] §3 Cor 1.

**Theorem 4.1** ([19] V.2.4.5). *The following hold:*

- (a) *every projective object in  $\mathfrak{C}(\Lambda)$  is isomorphic to a direct product  $\prod_{i \in I} P_i$ , for some set  $I$ , with  $P_i$  projective indecomposable objects;*
- (b) *with the notations of (a), if  $Q$  is a projective object in  $\mathfrak{C}(\Lambda)$  and  $q : \prod_{i \in I} P_i \twoheadrightarrow Q$  is an epimorphism then there exists a subset  $J$  of  $I$  such that  $q_J : \prod_{i \in I} P_i \rightarrow Q \oplus (\prod_{j \in J} P_j)$  induced by  $q$  and the canonical projections  $\prod_{i \in I} P_i \rightarrow P_j$  is an isomorphism;*
- (c) *suppose that  $\prod_{i \in I} P_i \cong \prod_{j \in J} Q_j$  with  $Q_j$  projective indecomposable then there exists a bijection  $h : I \rightarrow J$  such that  $P_i \cong Q_{h(i)}$  for all  $i$ .*

**Proposition 4.2.** *Let  $\mathcal{P}$  be an open normal pro- $p$  subgroup of  $\mathcal{G}$ , then the irreducible  $\Lambda$ -modules, coincide with the irreducible  $k[\mathcal{G}/\mathcal{P}]$ -modules. In particular, there are only finitely many irreducibles, and as  $A$ -modules they are finite dimensional  $k$ -vector spaces. Moreover, there exists an isomorphism of  $\Lambda$ -modules:*

$$\Lambda \cong \bigoplus_S (\dim_k S) P_S,$$

*where the direct sum is taken over all irreducible  $\Lambda$ -modules and  $P_S$  is a projective envelope of  $S$  in  $\mathfrak{C}(\Lambda)$ .*

*Proof.* Let  $S$  be an irreducible  $\Lambda$ -module in  $\mathfrak{C}(\Lambda)$ . The anti-equivalence of categories implies that  $S^\vee := \text{Hom}_A^{\text{cont}}(S, L/A)$  is an irreducible discrete  $p$ -torsion module of  $\mathcal{G}$ . Hence, the  $A$ -module structure of  $S^\vee$  is just a  $k$ -vector

space, and since  $S^\vee$  is discrete and  $\mathcal{P}$  is pro- $p$ , the subspace of  $\mathcal{P}$ -invariants of  $S^\vee$  is non-zero. Since  $S^\vee$  is irreducible we obtain that  $\mathcal{P}$  acts trivially on  $S^\vee$ . By dualizing back we get the result. Moreover, we have

$$\mathrm{Hom}_\Lambda(\Lambda, S) \cong \mathrm{Hom}_{k[\mathcal{G}/\mathcal{P}]}(k[\mathcal{G}/\mathcal{P}], S) \cong S.$$

This implies that

$$\Lambda / \mathrm{rad}(\Lambda) \cong \bigoplus_S (\dim_k S) S,$$

and since a projective module in  $\mathfrak{C}(\Lambda)$  is determined by its head, we get that

$$\Lambda \cong \bigoplus_S (\dim_k S) P_S.$$

□

From now on we assume that  $\mathcal{G}$  is a  $p$ -adic Lie group. Then it follows from results of Lazard [27] that  $\Lambda$  is noetherian, see [38] Cor. 2.4. Hence, the category of finitely generated modules  $\mathrm{Mod}_{fg}(\Lambda)$  is abelian. Moreover, if  $M$  is finitely generated over  $\Lambda$  then there exists a unique Hausdorff topology on  $M$  such that  $M$  is a topological  $\Lambda$ -module, and every  $\Lambda$ -homomorphism between finitely generated modules is continuous for the canonical topology, [36] Prop 3.1 or [29] §2 Prop 5.2.22. Hence, we may view  $\mathrm{Mod}_{fg}(\Lambda)$  as a full subcategory of  $\mathfrak{C}(\Lambda)$ .

**Proposition 4.3.** *Every object in  $\mathrm{Mod}_{fg}(\Lambda)$  has a projective envelope. Moreover, Theorem 4.1 holds for  $\mathrm{Mod}_{fg}(\Lambda)$  if we replace products with finite direct sums.*

*Proof.* A projective indecomposable object in  $\mathfrak{C}(\Lambda)$  is a direct summand of  $\Lambda$ , hence lies in  $\mathrm{Mod}_{fg}(\Lambda)$ . Let  $M$  be a finitely generated  $\Lambda$ -module. Then there exists a surjection  $\alpha : \Lambda^n \twoheadrightarrow M$ , for some integer  $n$ . Let  $\beta : P \twoheadrightarrow M$  be a projective envelope of  $M$  in  $\mathfrak{C}(\Lambda)$ . Since  $\Lambda^n$  is projective there exists  $\gamma : \Lambda^n \rightarrow P$  such that  $\alpha = \beta \circ \gamma$ . Since  $\beta$  is essential,  $\gamma$  is surjective, and since  $P$  is projective,  $\gamma$  has a section. So  $P$  is isomorphic to a direct summand of  $\Lambda^n$ . Proposition 4.2 and Theorem 4.1 implies that  $P$  is isomorphic to a finite direct sum of projective indecomposable modules. Hence  $P$  is finitely generated. The same argument with  $P = M$  gives that every finitely generated projective module in  $\mathfrak{C}(\Lambda)$  is isomorphic to a finite direct sum of indecomposable projective modules. The last assertion follows from Theorem 4.1. □

**Lemma 4.4.** *Let  $S$  be an irreducible  $A[[\mathcal{G}]]$ -module and let  $P_S$  be a projective envelope of  $S$  in  $\text{Mod}_{fg}(A[[\mathcal{G}]])$  then  $P_S \otimes_A k$  is a projective envelope of  $S$  in  $\text{Mod}_{fg}(k[[\mathcal{G}]])$ .*

*Proof.* Since as  $A$ -module  $S$  is a  $k$ -vector space the map  $P_S \rightarrow S$  factors through  $P_S \otimes_A k$ . This implies that  $P_S \otimes_A k \rightarrow S$  is an essential epimorphism. Since  $P_S$  is isomorphic to a direct summand of  $A[[\mathcal{G}]]$ ,  $P_S \otimes_A k$  is isomorphic to a direct summand of  $k[[\mathcal{G}]]$ , hence it is projective.  $\square$

**Proposition 4.5.** *Let  $P$  and  $M$  be finitely generated  $A[[\mathcal{G}]]$ -module. Suppose that  $P$  is projective and we have two surjective homomorphisms of  $A[[\mathcal{G}]]$ -modules  $\psi_1 : P \rightarrow M$ ,  $\psi_2 : P \rightarrow M$ . Then there exists  $\phi \in \text{Aut}_{A[[\mathcal{G}]]}(P)$  such that  $\psi_2 \circ \phi = \psi_1$ .*

*Proof.* Let  $P_M$  be a projective envelope of  $M$  in  $\text{Mod}_{fg}(A[[\mathcal{G}]])$ . Since  $\psi_i$  are surjective, for  $i \in \{0, 1\}$  there exists idempotents  $e_i \in \text{End}_{A[[\mathcal{G}]]}(P)$  such that  $(1 - e_i)P$  lies in the kernel of  $\psi_i$ ,  $e_i P \cong P_M$  and  $\psi_i : e_i P \rightarrow M$  is a projective envelope of  $M$ . Since projective envelopes are unique up to isomorphism, there exists an isomorphism of  $\phi_1 : e_1 P \cong e_2 P$ , such that  $\psi_2 \circ \phi_1 = \psi_1$ . It follow from Theorem 4.1 (c) that there exists an isomorphism of  $A[[\mathcal{G}]]$ -modules  $\phi_2 : (1 - e_1)P \cong (1 - e_2)P$ . Let  $\phi := (\phi_1, \phi_2)$  be a homomorphism  $P = e_1 P \oplus (1 - e_1)P \rightarrow e_2 P \oplus (1 - e_2)P = P$ . Then  $\phi$  is an isomorphism and  $\psi_2 \circ \phi = \psi_1$ .  $\square$

**Proposition 4.6.** *Let  $M$  be a finitely generated  $A[[\mathcal{G}]]$ -module, which is  $A$ -torsion free. Assume that  $M \otimes_A k$  is a projective object in  $\mathfrak{C}(k[[\mathcal{G}]])$ . Then*

$$M \otimes_A k \cong \bigoplus_S m_S (P_S \otimes_A k), \quad M \cong \bigoplus_S m_S P_S,$$

where the sum is taken over irreducible modules  $S$ ,  $m_S$  denotes some finite multiplicities, and  $P_S$  denotes a projective envelope of  $S$  in  $\mathfrak{C}(A[[\mathcal{G}]])$ .

*Proof.* Since  $M$  is finitely generated over  $A[[\mathcal{G}]]$ ,  $M \otimes_A k$  is finitely generated over  $k[[\mathcal{G}]]$ . Proposition 4.3 implies that there exists uniquely determined non-negative integers  $m_S$ , such that  $M \otimes_A k \cong \bigoplus_S m_S \overline{P}_S$ , where  $\overline{P}_S$  is a projective envelope of  $S$  in  $\mathfrak{C}(k[[\mathcal{G}]])$ . Lemma 4.4 implies that there exists an isomorphism  $P_S \otimes_A k \cong \overline{P}_S$ . Set  $P := \bigoplus_S m_S P_S$ . Since  $P$  is projective there exists  $\psi : P \rightarrow M$  making the following diagram commute:

$$\begin{array}{ccc} P & \xrightarrow{\psi} & M \\ \downarrow & & \downarrow \\ P \otimes_A k & \xrightarrow{\cong} & M \otimes_A k \end{array} \tag{3}$$

Let  $Q$  be the cokernel of  $\psi$ . Then  $Q \otimes_A k = 0$ , and since  $M$  is a compact  $A$ -module,  $Q$  is a compact  $A$ -module. Nakayama's lemma [20] Exp. VII<sub>B</sub> (0.3.3) implies that  $Q = 0$ . Hence  $\psi$  is surjective. Since  $M$  is  $A$ -torsion free, it is a flat  $A$ -module. This implies that  $(\text{Ker } \psi) \otimes_A k = 0$ , which again by Nakayama's lemma gives  $\text{Ker } \psi = 0$ .  $\square$

**Corollary 4.7.** *Let  $P$  be a finitely generated projective  $A[[\mathcal{G}]]$ -module then the reduction map  $\text{Aut}_{A[[\mathcal{G}]]}(P) \rightarrow \text{Aut}_{k[[\mathcal{G}]]}(P \otimes_A k)$  is surjective.*

*Proof.* Set  $M = P$  in the diagram (3).  $\square$

**Proposition 4.8.** *Let  $S$  be an irreducible  $A[[\mathcal{G}]]$ -module and let  $P_S$  be a projective envelope of  $S$ , let  $M$  be an  $A[[\mathcal{G}]]$ -module, such that  $M$  as an  $A$ -module is free of finite rank. Then  $\text{Hom}_{A[[\mathcal{G}]]}(P_S, M)$  is a free  $A$ -module of rank  $m$ , where  $m$  is the multiplicity with which  $S$  occurs as a subquotient of  $\overline{M}$ .*

*Proof.* We claim that  $\text{Hom}_{A[[\mathcal{G}]]}(P_S, \overline{M})$  is a  $k$ -vector space of dimension  $m$ . Since  $M$  is a free  $A$ -module of finite rank,  $\overline{M}$  is a finite dimensional  $k$ -vector space. In particular  $\overline{M}$  is an  $A[[\mathcal{G}]]$ -module of finite length. Suppose that  $\overline{M}$  is irreducible, then since  $P_S$  is a projective envelope of  $S$ , we have  $\dim_k \text{Hom}_{A[[\mathcal{G}]]}(P_S, \overline{M}) = 1$  if  $S \cong \overline{M}$ , and  $\dim_k \text{Hom}_{A[[\mathcal{G}]]}(P_S, \overline{M}) = 0$  if  $S \not\cong \overline{M}$ . In general, let  $S'$  be an irreducible submodule of  $\overline{M}$ . Since  $P_S$  is projective we get an exact sequence:

$$0 \rightarrow \text{Hom}_{A[[\mathcal{G}]]}(P_S, S') \rightarrow \text{Hom}_{A[[\mathcal{G}]]}(P_S, \overline{M}) \rightarrow \text{Hom}_{A[[\mathcal{G}]]}(P_S, \overline{M}/S') \rightarrow 0.$$

We get the assertion by induction on the length of  $\overline{M}$ . Now  $\text{Hom}_{A[[\mathcal{G}]]}(P_S, M)$  is a direct summand of  $\text{Hom}_{A[[\mathcal{G}]]}(A[[\mathcal{G}]], M) \cong M$ . Hence,  $\text{Hom}_{A[[\mathcal{G}]]}(P_S, M)$  is a free  $A$ -module of finite rank. We have

$$\text{Hom}_{A[[\mathcal{G}]]}(P_S, M) \otimes_A k \cong \text{Hom}_{A[[\mathcal{G}]]}(P_S, M \otimes_A k) \cong k^m.$$

Hence,  $\text{Hom}_{A[[\mathcal{G}]]}(P_S, M)$  is a free  $A$ -module of rank  $m$ .  $\square$

We set  $L[[\mathcal{G}]] := A[[\mathcal{G}]] \otimes_A L$ , as  $A[[\mathcal{G}]]$  is noetherian, so is  $L[[\mathcal{G}]]$ . Hence the category  $\text{Mod}_{fg}(L[[\mathcal{G}]])$  of finitely generated  $L[[\mathcal{G}]]$ -modules is abelian.

**Corollary 4.9.** *Let  $S$  be an irreducible  $A[[\mathcal{G}]]$ -module, and let  $P_S$  be a projective envelope of  $S$  in  $\text{Mod}_{fg}(A[[\mathcal{G}]])$ . Let  $V$  be an  $L[[\mathcal{G}]]$ -module, such that  $V$  is a finite dimensional  $L$ -vector space. Let  $M$  be any  $\mathcal{G}$ -invariant  $A$ -lattice in  $V$  then  $\dim_L \text{Hom}_{L[[\mathcal{G}]]}(P_S \otimes_A L, V) = m$ , where  $m$  is the multiplicity with which  $S$  occurs in  $\overline{M}$ .*

*Proof.* It follows from the discussion in [36] before Proposition 3.1, that

$$\mathrm{Hom}_{L[[\mathcal{G}]]}(P_S \otimes_A L, V) \cong \mathrm{Hom}_{A[[\mathcal{G}]]}(P_S, M) \otimes_A L.$$

The assertion follows from 4.8.  $\square$

## 5 Banach space representations

Let  $\mathcal{G}$  be a compact  $p$ -adic Lie group. We recall some facts about Banach space representations of  $\mathcal{G}$ . We follow closely Schneider-Teitelbaum [36]. Let  $\mathrm{Ban}_L$  denote the category of  $L$ -Banach spaces. We note that we do not fix a norm defining the topology on the Banach space  $E$ , when we do want to fix such a norm  $\|\cdot\|$  we will write  $(E, \|\cdot\|)$ .

**Definition 5.1.** *An  $L$ -Banach space representation  $E$  of  $\mathcal{G}$  is an  $L$ -Banach space  $E$  together with a  $\mathcal{G}$ -action by continuous linear automorphisms such that the map  $\mathcal{G} \times E \rightarrow E$  describing the action is continuous.*

Let  $\mathrm{Ban}_L(\mathcal{G})$  be the category of  $L$ -Banach space representations with morphisms being all  $\mathcal{G}$ -equivariant continuous linear maps.

**Definition 5.2.** *An  $L$ -Banach space representation  $E$  of  $\mathcal{G}$  is called admissible if there exists a  $\mathcal{G}$ -invariant bounded open  $A$ -submodule  $M \subseteq E$  such that for any open pro- $p$  group  $\mathcal{P}$  of  $\mathcal{G}$ , the  $A$ -submodule of  $(E/M)^\mathcal{P}$  is of cofinite type (i.e.  $\mathrm{Hom}_A((E/M)^\mathcal{P}, L/A)$  is a finitely generated  $A$ -module).*

Let  $\mathrm{Ban}_L^{\mathrm{adm}}(\mathcal{G})$  be the full subcategory of  $\mathrm{Ban}_L(\mathcal{G})$  consisting of admissible  $L$ -Banach representations of  $\mathcal{G}$ . It follows from [36] Theorem 3.5, that  $\mathrm{Ban}_L^{\mathrm{adm}}(\mathcal{G})$  is an abelian category.

Recall that a topological  $A$ -module  $M$  is linearly topological if 0 has a fundamental system of open neighbourhoods consisting of  $A$ -submodules. Let  $\mathrm{Mod}_{\mathrm{top}}(A)$  be the category of all Hausdorff linearly topological  $A$ -modules, with morphisms all continuous  $A$ -linear maps. Let  $\mathrm{Mod}_{\mathrm{comp}}^{\mathrm{fl}}(A)$  be the full subcategory of  $\mathrm{Mod}_{\mathrm{top}}(A)$  consisting of all torsion-free and compact linearly topological  $A$ -modules. The superscript  $\mathrm{fl}$  stands for flat. Following [36] we recall that an  $A$ -module is torsion-free if and only if it is flat, [7, I §2.4 Prop 3(ii)]; a compact linear-topological  $A$ -module  $M$  is flat if and only if  $M \cong \prod_{i \in I} A$  for some set  $I$ , [20, VII<sub>B</sub> (0.3.8)]. Given  $M$  in  $\mathrm{Mod}_{\mathrm{comp}}^{\mathrm{fl}}(A)$  we define

$$M^d := \mathrm{Hom}_A^{\mathrm{cont}}(M, L).$$



Now  $M^d$  carries a structure of an  $L$ -Banach space with  $\|\ell\| := \max_{v \in M} |\ell(v)|$ . Let  $\text{Mod}_{fg}^{fl}(A[[\mathcal{G}]])$  be the category of finitely generated  $A[[\mathcal{G}]]$ -modules, which are  $A$ -torsion free. Given  $M$  in  $\text{Mod}_{fg}^{fl}(A[[\mathcal{G}]])$  we equip it with the canonical topology, then  $M$  is an object in  $\text{Mod}_{comp}^{fl}(A)$ . Given an additive category  $\mathfrak{A}$  we denote by  $\mathfrak{A}_{\mathbb{Q}}$  the additive category with the same objects as  $\mathfrak{A}$  and  $\text{Hom}_{\mathfrak{A}_{\mathbb{Q}}}(A, B) := \text{Hom}_{\mathfrak{A}}(A, B) \otimes \mathbb{Q}$ .

**Theorem 5.3** ([36], Thm 1.2, Thm. 3.5). *The functor  $M \mapsto M^d$  induces an anti-equivalence of categories*

$$\text{Mod}_{comp}^{fl}(A)_{\mathbb{Q}} \xrightarrow{\sim} \text{Ban}_L, \quad \text{Mod}_{fg}^{fl}(A[[\mathcal{G}]])_{\mathbb{Q}} \xrightarrow{\sim} \text{Ban}_{\mathcal{G}}^{adm}(L).$$

Note that  $\text{Mod}_{fg}^{fl}(A[[\mathcal{G}]])_{\mathbb{Q}}$  is equivalent to  $\text{Mod}_{fg}(L[[\mathcal{G}]])$ . Let  $E$  be an  $L$ -Banach space the object  $E^d$  in  $\text{Mod}_{comp}^{fl}(A)_{\mathbb{Q}}$  corresponding to  $E$  is constructed as follows, see the proof of [36, Thm 1.2]. We may choose a norm  $\|\cdot\|$  defining the topology on  $E$ , and such that  $\|E\| \subseteq |L|$ . Let  $E^0$  be the unit ball in  $E$  with respect to  $\|\cdot\|$ , set

$$E^d := \text{Hom}_A(E^0, A),$$

with the topology of pointwise convergence, that is the coarsest locally convex topology such that for each  $v \in E^0$  the map  $E^d \rightarrow A, \phi \mapsto \phi(v)$  is continuous.

**Lemma 5.4.** *Let  $M$  be in  $\text{Mod}_{comp}^{fl}(A)$  and let  $(M^d)^0$  be the unit ball in  $M^d$  with respect to the supremum norm then there exists a canonical isomorphism  $(M^d)^0 \otimes_A k \cong (M \otimes_A k)^{\vee}$ . Conversely let  $(E, \|\cdot\|)$  be an  $L$ -Banach space, assume that  $\|E\| \subseteq |L|$ . Let  $E^0$  be the unit ball in  $E$ , and let  $M := \text{Hom}_A(E^0, A)$  with the topology as above. Then there exists a canonical topological isomorphism  $M \otimes_A k \cong (E^0 \otimes_A k)^{\vee}$ .*

*Proof.* The reduction map  $A \rightarrow k$  is continuous. Hence, we obtain a homomorphism of  $A$ -modules  $r : \text{Hom}_A^{cont}(M, A) \rightarrow \text{Hom}_A^{cont}(M, k)$ . We claim that  $r$  is surjective. We note that the claim is clear if  $M$  is of finite rank. In general  $M \cong \prod_{i \in I} A$ , for some set  $I$ . So if  $\phi \in \text{Hom}_A^{cont}(M, k)$  then there exists a subset  $J \subseteq I$  with  $I \setminus J$  finite and an integer  $n \geq 1$ , such that  $\prod_{j \in J} A \times \prod_{i \in I \setminus J} \mathfrak{M}^n$  is contained in the kernel of  $\phi$ , since such subsets form a basis of open neighbourhoods of 0 in  $\prod_{i \in I} A$ . The problem reduces to showing that the map  $\text{Hom}_A^{cont}(\prod_{i \in I \setminus J} A, A) \rightarrow \text{Hom}_A^{cont}(\prod_{i \in I \setminus J} A, k)$  is surjective. Since  $I \setminus J$  is finite we are done. The claim yields a short exact sequence of  $A$ -modules:

$$0 \rightarrow \text{Hom}_A^{cont}(M, A) \xrightarrow{\varpi_I} \text{Hom}_A^{cont}(M, A) \rightarrow \text{Hom}_A^{cont}(M, k) \rightarrow 0. \quad (4)$$

On the other hand,  $\text{Hom}_A^{\text{cont}}(M, A)$  is torsion-free and hence flat. So tensoring the short exact sequence  $0 \rightarrow A \xrightarrow{\varpi_L} A \rightarrow k \rightarrow 0$  with  $\text{Hom}_A^{\text{cont}}(M, A)$  we obtain a short exact sequence:

$$0 \rightarrow \text{Hom}_A^{\text{cont}}(M, A) \xrightarrow{\varpi_L} \text{Hom}_A^{\text{cont}}(M, A) \rightarrow \text{Hom}_A^{\text{cont}}(M, A) \otimes_A k \rightarrow 0. \quad (5)$$

Now, (4) and (5) imply that the natural map

$$\text{Hom}_A^{\text{cont}}(M, A) \otimes_A k \rightarrow \text{Hom}_A^{\text{cont}}(M, k)$$

is an isomorphism. Hence,  $(M^d)^0 \otimes_A k \cong \text{Hom}_A^{\text{cont}}(M \otimes_A k, k) \cong (M \otimes_A k)^\vee$ ;  $(M^d)^0 \otimes_A k$  carries the discrete topology,  $M \otimes_A k$  is compact and so  $(M \otimes_A k)^\vee$  also carries the discrete topology. The second part follows from Theorem 5.3.  $\square$

**Lemma 5.5.** *Let  $(E, \|\cdot\|)$  be an  $L$ -Banach space, such that  $\|E\| \subseteq |L|$ . Let  $(E_1, \|\cdot\|)$  be a closed subspace. Then we have an exact sequence of  $A$ -modules:*

$$0 \rightarrow E_1^0 \rightarrow E^0 \rightarrow (E/E_1)^0 \rightarrow 0, \quad (6)$$

$$0 \rightarrow E_1^0 \otimes_A k \rightarrow E^0 \otimes_A k \rightarrow (E/E_1)^0 \otimes_A k \rightarrow 0, \quad (7)$$

where superscript 0 denotes the unit ball in the respective Banach space.

*Proof.* The quotient space  $E/E_1$  carries a norm defined by

$$\|v + E_1\| := \inf_{u \in E_1} \|v + u\|, \quad \forall v \in E.$$

It is clear that  $E^0$  maps into  $(E/E_1)^0$ . Since  $L$  is discretely valued for every  $v \in E$  there exists  $u \in E_1$  such that  $\|v + E_1\| = \|v + u\|$ . Hence, we obtain a surjection  $E^0 \twoheadrightarrow (E/E_1)^0$ . This implies (6). Now  $(E/E_1)^0$  is torsion-free and hence flat. By tensoring (6) with  $\otimes_A k$  we obtain (7).  $\square$

Let  $V$  be an  $L$ -vector space and  $M$  an  $A$ -submodule of  $V$ . We say that  $M$  is a *lattice* in  $V$ , if for every  $v \in V$  there exists  $x \in L^\times$  such that  $xv \in M$ , [34] §2. We say that  $M$  is *separated* if  $\bigcap_{n \geq 0} \varpi_L^n M = 0$ . If  $E$  is an  $L$ -Banach space and  $M$  is an open lattice in  $E$  is separated if and only if it is bounded. Moreover, if  $M$  is an open separated lattice in  $E$  then the gauge of  $M$ , [34] §2, defined by

$$\|v\|_M := \inf_{v \in aM} |a|, \quad \forall v \in E$$

is a norm (since  $M$  is separated), and the topology on  $E$  defined by  $\|\cdot\|_M$  coincides with the original one (since  $M$  is open). If  $E$  is an  $L$ -Banach space

representation of  $\mathcal{G}$  then (since  $\mathcal{G}$  is compact) there exists an open separated  $\mathcal{G}$ -invariant lattice  $M$  in  $E$ , [22] Lemma 6.5.5, [23] Example 3.7, Lemma 3.9. Since  $M$  is  $\mathcal{G}$ -invariant we have  $\|gv\|_M = \|v\|_M$  for all  $v \in M$  and  $g \in \mathcal{G}$ , so  $E$  is a *unitary*  $L$ -Banach space representation of  $\mathcal{G}$ . Since  $L$  is discretely valued  $\|E\|_M \subseteq |L|$ .

**Proposition 5.6.** *Let  $(E, \|\cdot\|)$  be a unitary  $L$ -Banach space representation of  $\mathcal{G}$ , such that  $\|E\| \subseteq |L|$ . Let  $E^0$  be the unit ball in  $E$ . Suppose that  $E^0 \otimes_A k \cong I$ , where  $I$  is an injective admissible object in  $\text{Rep}_k(\mathcal{G})$ . Let  $m_S := \dim_k \text{Hom}_{\mathcal{G}}(S, I)$  then there exists a  $\mathcal{G}$ -equivariant isometrical isomorphism*

$$(E, \|\cdot\|) \xrightarrow{\cong} \bigoplus_{S \in \text{Irr}(\mathcal{G})} (P_{S^*}^d)^{\oplus m_S},$$

where  $P_{S^*}$  is the projective envelope of  $S^*$  in  $\mathfrak{C}(A[[\mathcal{G}]])$ , and the right hand-side is equipped with the supremum norm.

In particular, if  $\mathcal{P}$  is an open pro- $p$  subgroup of  $\mathcal{G}$  and  $m := \dim_k I^{\mathcal{P}}$  then there exists a  $\mathcal{P}$ -equivariant isometrical isomorphism

$$(E, \|\cdot\|) \xrightarrow{\cong} C(\mathcal{P}, L)^{\oplus m},$$

where  $C(\mathcal{P}, L)$  denotes the space of continuous functions from  $\mathcal{P}$  to  $L$  with the supremum norm.

*Proof.* Since  $E^0 \otimes_A k$  is admissible in  $\text{Rep}_k(\mathcal{G})$ , it follows from [22] Proposition 6.5.7 that  $E$  is an admissible  $L$ -Banach space representation of  $\mathcal{G}$  in the sense of Definition 5.2. Set  $M := \text{Hom}_A(E^0, A)$ , then Lemma 5.4 implies that  $M \otimes_A k \cong I^{\vee}$ . It follows from Lemma 3.6 that  $I \cong \bigoplus_S I_S^{\oplus m_S}$ . Since  $I_S$  is injective  $I_S^{\vee}$  is a projective  $k[[\mathcal{G}]]$ -module. Since  $I_S$  is admissible  $I_S^{\vee}$  is finitely generated over  $k[[\mathcal{G}]]$ , [40] or the proof of [22] Proposition 6.5.7. Since  $I_S$  is an injective envelope of  $S$ ,  $I_S^{\vee}$  is a projective envelope of  $S^*$  in  $\text{Mod}_{fg}(k[[\mathcal{G}]])$ . Proposition 4.6 implies that  $M \cong \bigoplus_S P_{S^*}^{\oplus m_S}$ . The assertion follows from Theorem 5.3.

If  $\mathcal{P}$  is an open pro- $p$  group of  $\mathcal{G}$  then  $I|_{\mathcal{P}}$  is an admissible injective object in  $\text{Rep}_k(\mathcal{P})$ . Moreover, since  $\mathcal{P}$  is a pro- $p$  group the only irreducible representation of  $\mathcal{P}$  is the trivial one  $\mathbf{1}$ . And  $m_{\mathbf{1}} = \dim_k \text{Hom}_{\mathcal{G}}(\mathbf{1}, I) = m$ . The space of continuous functions  $C(\mathcal{P}, k)$  from  $\mathcal{P}$  to  $k$  is an injective envelope of  $\mathbf{1}$  in  $\text{Rep}_k(\mathcal{P})$ , Lemma 3.7. So  $I|_{\mathcal{P}} \cong C(\mathcal{P}, k)^{\oplus m}$  hence  $I^{\vee}|_{\mathcal{P}} \cong k[[\mathcal{P}]]^{\oplus m}$  as a  $k[[\mathcal{P}]]$ -module. This implies that  $M \cong A[[\mathcal{P}]]^{\oplus m}$ . It follows from [36] Lemma 2.1, Corollary 2.2 that  $(A[[\mathcal{P}]])^d \cong C(\mathcal{P}, L)$ .  $\square$

**Corollary 5.7.** *Let  $I$  be an admissible injective object in  $\text{Rep}_k(\mathcal{G})$  then there exists an admissible unitary  $L$ -Banach space representation  $(E, \|\cdot\|)$ , such that  $E^0 \otimes_A k \cong I$ , where  $E^0$  is the unit ball in  $E$ .*

*Proof.* Let  $P$  be a finitely generated projective object in  $\mathfrak{C}(A[[\mathcal{G}]])$  such that  $P \otimes_A k \cong I^\vee$ . Set  $E := P^d$ , with the supremum norm. Lemma 5.4 implies the assertion.  $\square$

**Lemma 5.8.** *Let  $E$  be an admissible  $L$ -Banach space representation of  $\mathcal{G}$ , then any decreasing sequence of closed  $\mathcal{G}$ -invariant subspaces becomes constant.*

*Proof.* Suppose that  $E_i \subseteq E$  is a closed  $\mathcal{G}$ -invariant subspace. Dually we have a surjection of  $L[[\mathcal{G}]]$ -modules  $E^d \twoheadrightarrow E_i^d$ , recall that  $\text{Mod}_{fg}^{fl}(A[[\mathcal{G}]])_{\mathbb{Q}}$  is equivalent to  $\text{Mod}_{fg}(L[[\mathcal{G}]])$ . Let  $M_i$  denote the kernel, then we obtain an increasing sequence of  $L[[\mathcal{G}]]$ -submodules of  $E^d$ ,  $M_1 \subseteq M_2 \subseteq \dots \subseteq E^d$ . Since  $E$  is admissible  $E^d$  is finitely generated, and since  $L[[\mathcal{G}]]$  is noetherian there exists  $m$  such that  $M_i = M_m$  for all  $i \geq m$ . Hence  $E_i^d = E_m^d$  and so  $E_i = E_m$  for all  $i \geq m$ .  $\square$

## 6 Lifting $\Omega$

We assume throughout that  $p \neq 2$ . Let  $G := \text{GL}_2(F)$ ,  $Z$  the centre of  $G$ ,  $K := \text{GL}_2(\mathfrak{o})$ ,

$$I := \begin{pmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o}^\times \end{pmatrix}, \quad I_1 := \begin{pmatrix} 1 + \mathfrak{p} & \mathfrak{o} \\ \mathfrak{p} & 1 + \mathfrak{p} \end{pmatrix}.$$

Let  $\mathfrak{K}_0$  be the  $G$ -normalizer of  $K$ , and  $\mathfrak{K}_1$  be the  $G$ -normalizer of  $I$ , then  $\mathfrak{K}_0 = KZ$  and  $\mathfrak{K}_1$  is generated as a group by  $I$  and the element  $\Pi := \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}$ . We fix a uniformizer  $\varpi$  of  $F$ , and consider as an element of  $Z$ , via  $Z \cong F^\times$ .

**Theorem 6.1.** *Let  $\Omega \in \text{Rep}_k(G)$  be such that  $\varpi$  acts trivially,  $\Omega|_K$  is an injective admissible object in  $\text{Rep}_k(K)$ . Then there exists a unitary admissible  $L$ -Banach space representation  $(E, \|\cdot\|)$  such that*

$$E^0 \otimes_A k \cong \Omega,$$

*as  $G$ -representations, where  $E^0$  denotes the unit ball in  $E$ , with respect to  $\|\cdot\|$ .*

*Proof.* Since  $\Omega|_K$  is injective in  $\text{Rep}_k(K)$ ,  $\Omega|_I$  is injective in  $\text{Rep}_k(I)$ . Since  $\varpi$  acts trivially on  $\Omega$ , we may consider  $\Omega$  as a representation of  $\mathcal{G} := \mathfrak{K}_1/\varpi^\mathbb{Z}$ . The index of  $I$  in  $\mathcal{G}$  is 2, and since  $p \neq 2$  by assumption, we get that the maximal pro- $p$  subgroup of  $\mathcal{G}$  is contained in  $I$ . This implies that  $\Omega|_{\mathcal{G}}$  is an injective representation of  $\mathcal{G}$ . Corollary 5.7 applied to  $(\mathcal{G}, \Omega)$ , gives an admissible unitary  $L$ -Banach space representation  $(E_1, \|\cdot\|_1)$  of  $\mathcal{G}$  such that we have an isomorphism of  $\mathcal{G}$ -representations  $\iota_1 : E_1^0 \otimes_A k \cong \Omega$ . We let  $\varpi$  act trivially on  $E_1$ , so that  $\iota_1$  is  $\mathfrak{K}_1$ -equivariant. Corollary 5.7 applied to  $(K, \Omega)$  gives an admissible unitary  $L$ -Banach space representation  $(E_0, \|\cdot\|_0)$  of  $K$  such that we have an isomorphism of  $\mathcal{G}$ -representations  $\iota_0 : E_0^0 \otimes_A k \cong \Omega$ . We let  $\varpi$  act trivially on  $E_0$ , so that  $\iota_0$  is  $\mathfrak{K}_0$ -equivariant. Now  $\iota_0^{-1} \circ \iota_1$  induces an  $ZI$ -equivariant isomorphism  $E_1^0 \otimes_A k \cong E_0^0 \otimes_A k$ . It follows from Corollary 4.7 that there exists a  $ZI$ -equivariant isometrical isomorphism:

$$\phi : (E_1, \|\cdot\|_1) \cong (E_0, \|\cdot\|_0),$$

such that  $\phi \otimes 1 = \iota_0^{-1} \circ \iota_1$ . We may transport the action of  $\mathfrak{K}_1$  on  $(E_0, \|\cdot\|_0)$ , by setting

$$g \cdot v = \phi(g\phi^{-1}(v)), \quad \forall v \in E_0, \forall g \in \mathfrak{K}_1.$$

If we restrict to  $I\mathbb{Z} = \mathfrak{K}_0 \cap \mathfrak{K}_1$  the two actions coincide, since  $\phi$  is  $I\mathbb{Z}$ -equivariant. Since  $G$  is an amalgam of  $\mathfrak{K}_0$  and  $\mathfrak{K}_1$  along  $I\mathbb{Z}$ , the two actions glue to an action of  $G$ . So we get an  $L$ -Banach space representation of  $G$  on  $(E, \|\cdot\|)$ , which is unitary, since it is unitary for the actions of  $\mathfrak{K}_0$  and  $\mathfrak{K}_1$ . By construction we obtain a  $G$ -equivariant isomorphism  $E^0 \otimes_A k \cong \Omega$ . Instead of using the amalgamation, one could also argue formally as in [30, Cor. 5.5.5].  $\square$

We note that although the lifts  $(E_i, \|\cdot\|_i)$  are unique up to  $\mathfrak{K}_i$ -equivariant isometry, there is no unique way to choose  $\phi$ , so the Banach space representation  $(E, \|\cdot\|)$  is not canonical. Moreover, it is enough to assume that  $\varpi$  acts by a scalar on  $\Omega$ , since after twisting by an unramified character we may get to the situation of Theorem 6.1. The following is a Banach space analog of [12, Cor. 9.11].

**Proposition 6.2.** *Let  $(E_1, \|\cdot\|_1)$  be an admissible unitary  $L$ -Banach space representation of  $G$ , such that  $\varpi$  acts trivially and  $\|E_1\|_1 \subseteq |L|$ . Let  $\sigma$  be the  $K$ -socle of  $E_1^0 \otimes_A k$ , then there exists a unitary  $L$ -Banach space representation  $(E, \|\cdot\|)$  of  $G$ , such that the restriction of  $E^0 \otimes_A k$  to  $K$  is an injective envelope of  $\sigma$  in  $\text{Rep}_k(K)$  and a  $G$ -equivariant isometry  $(E_1, \|\cdot\|_1) \hookrightarrow (E, \|\cdot\|)$ .*

*Proof.* Since  $E_1$  is an admissible Banach space representation,  $E_1^0 \otimes_A k$  is an admissible smooth  $k$ -representation of  $G$ . By [12, Cor. 9.11] there exists a  $G$ -equivariant embedding  $\iota : E_1^0 \otimes_A k \hookrightarrow \Omega$ , where  $\Omega$  is a smooth  $k$ -representation

of  $G$ , such that  $\Omega|_K$  is an injective envelope of  $\sigma$  in  $\text{Rep}_k(K)$ . Let  $(E, \|\cdot\|)$  be a lift of  $\Omega$ , given by Theorem 6.1. For  $i \in \{0, 1\}$  set  $\mathcal{G}_i := \mathfrak{K}_i/\varpi^{\mathbb{Z}}$  then dually, we have a diagram of  $A[[\mathcal{G}_i]]$ -modules:

$$\begin{array}{ccc} (E^0)^d & & (E_1^0)^d \\ \downarrow & & \downarrow \\ \Omega^\vee & \xrightarrow{\iota^\vee} & (E^0 \otimes_A k)^\vee \end{array}$$

Since  $\Omega$  is injective,  $\Omega^\vee$  is a projective  $k[[\mathcal{G}_i]]$ -module and so  $(E^0)^d$  is a projective  $A[[\mathcal{G}_i]]$ -module, Proposition 5.6. Hence there exists an  $A[[\mathcal{G}_i]]$ -module homomorphism  $\psi_i : (E_0)^d \rightarrow (E_1^0)^d$  making the diagram commute. Nakayama's Lemma implies that  $\psi_i$  is surjective, see the proof of Proposition 4.6. Since  $\Omega|_I$  is injective in  $\text{Rep}_k(I)$ ,  $(E^0)^d$  is a projective  $A[[I]]$ -module. Since  $\psi_1$  and  $\psi_2$  are also homomorphisms of  $A[[I]]$ -modules, Proposition 4.5 gives us  $\phi \in \text{Aut}_{A[[I]]}((E^0)^d)$  such that  $\psi_1 = \psi_2 \circ \phi$ . We let  $\varpi$  act trivially everywhere. Dually we get a  $\mathfrak{K}_0$ -equivariant isometry  $\psi_0^d : (E_1, \|\cdot\|) \hookrightarrow (E, \|\cdot\|)$ , a  $\mathfrak{K}_1$ -equivariant isometry  $\psi_1^d : (E_1, \|\cdot\|) \hookrightarrow (E, \|\cdot\|)$  and an  $I\mathbb{Z} = \mathfrak{K}_0 \cap \mathfrak{K}_1$ -equivariant isometrical isomorphism  $\phi^d : (E, \|\cdot\|) \cong (E, \|\cdot\|)$ , such that  $\psi_1^d = \phi^d \circ \psi_2^d$ . The data  $E|_{\mathfrak{K}_0}, E|_{\mathfrak{K}_1}, \phi^d$  glues to a new representation of  $G$ ,  $(E', \|\cdot\|)$  as in the proof of Theorem 6.1, and by construction  $E'|_K = E|_K$ . Moreover, the map  $\psi_0^d : E_1 \rightarrow E'$  is a  $G$ -equivariant isometry, since by construction it is  $\mathfrak{K}_0$  and  $\mathfrak{K}_1$ -equivariant isometry and these groups generate  $G$ .  $\square$

**Corollary 6.3.** *Let  $\kappa$  be an irreducible smooth admissible  $k$ -representation of  $G$ , such that  $\varpi$  acts trivially. Then there exists an admissible topologically irreducible unitary  $L$ -Banach space representation  $(E, \|\cdot\|)$ , such that  $\text{Hom}_G(\kappa, E^0 \otimes_A k) \neq 0$ .*

*Proof.* We may embed  $\kappa \hookrightarrow \Omega$ , where  $\Omega$  is a smooth  $k$ -representation of  $G$ , such that  $\Omega|_K$  is an injective envelope of  $\kappa$  in  $\text{Rep}_k(K)$ . Let  $(E', \|\cdot\|')$  be a unitary  $L$ -Banach space representation of  $G$  lifting  $\Omega$  as in Theorem 6.1. Since  $E'$  is admissible, Lemma 5.8 implies that  $E'$  has an irreducible subobject  $E$ . Lemma 5.5 gives an injection  $E^0 \otimes_A k \hookrightarrow \Omega$ . Since  $\Omega|_K$  is an injective envelope of  $\kappa$ , we get that  $\kappa \cap (E^0 \otimes_A k) \neq 0$ . Since  $\kappa$  is irreducible, it is contained in  $E^0 \otimes_A k$ .  $\square$

## 7 Admissible completions

We briefly recall the theory of types for  $\mathrm{GL}_2(F)$ . Traditionally the smooth representations of  $G$  are considered over the field of complex numbers, however, since the theory is algebraic in nature, we may consider it over any algebraically closed field of characteristic 0. In the following we take the coefficients to be  $\overline{L}$ , the algebraic closure of  $L$ . Let  $\mathfrak{R}$  be the category of smooth representations of  $G$  on  $\overline{L}$ -vector spaces, then  $\mathfrak{R}$  decomposes into a product of subcategories

$$\mathfrak{R} \cong \prod_{\mathfrak{s} \in \mathfrak{B}} \mathfrak{R}_{\mathfrak{s}},$$

where  $\mathfrak{B}$  is the set of inertial equivalence classes of supercuspidal representations of the Levy subgroups of  $G$ , see [6], [15]. Following [25, Def A.1.4.1] we say that an irreducible smooth  $\overline{L}$ -representation  $\tau$  is *typical* for the Bernstein component  $\mathfrak{R}_{\mathfrak{s}}$ , if for every irreducible object  $\pi$  in  $\mathfrak{R}$ ,  $\mathrm{Hom}_K(\tau, \pi) \neq 0$  implies that  $\pi$  lies in  $\mathfrak{R}_{\mathfrak{s}}$ . We say that  $\tau$  is a *type* for  $\mathfrak{R}_{\mathfrak{s}}$  if it is typical and  $\mathrm{Hom}_K(\tau, \pi) \neq 0$  for every irreducible object  $\pi$  in  $\mathfrak{R}_{\mathfrak{s}}$ . Given  $\mathfrak{R}_{\mathfrak{s}}$ , there exists a type  $\tau$ , unique up to isomorphism, except when  $\mathfrak{R}_{\mathfrak{s}}$  contains  $\chi \circ \det$ . In this case, there are two typical representations  $\theta \circ \det$  and  $\mathrm{St} \otimes \theta \circ \det$ , where  $\theta := \chi|_{\mathfrak{o}^\times}$  and  $\mathrm{St}$  is the lift to  $K$  of the Steinberg representations of  $\mathrm{GL}_2(\mathbb{F}_q)$ , see [25]. It follows from [25] that the definition of a type here coincides with the one given in [15, Def 4.1]. In particular, if  $\tau$  is a type for  $\mathfrak{R}_{\mathfrak{s}}$  and  $\pi$  is a smooth representation of  $G$ , with a  $K$ -invariant subspace  $W$  isomorphic to  $\tau$  then the subspace  $\langle G \cdot W \rangle$  of  $\pi$ , is an object of  $\mathfrak{R}_{\mathfrak{s}}$ .

Let  $\Sigma$  be the set of  $\mathbb{Q}_p$ -linear embeddings of fields  $F \hookrightarrow L$ . We choose  $L$  to be 'large', so that  $[F : \mathbb{Q}_p] = |\Sigma|$ . When needed we will replace  $L$  with some finite extension.

For us a  $\mathbb{Q}_p$ -rational representation of  $G$ , is a representation  $W$  of the form

$$\bigotimes_{\sigma \in \Sigma} (\mathrm{Sym}^{r_\sigma} L^2 \otimes \det^{a_\sigma})^\sigma,$$

where  $r_\sigma, a_\sigma$  are integers,  $r_\sigma \geq 0$ , and an element  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  in  $G$  acts on the  $\sigma$ -component via  $\begin{pmatrix} \sigma(a) & \sigma(b) \\ \sigma(c) & \sigma(d) \end{pmatrix}$ , see [11, §2] for a proper setting. Such  $W$  are absolutely irreducible and remain absolutely irreducible when restricted to an open subgroup of  $G$ , (this is used implicitly in [11, Lem 2.1], one may argue by Zariski density as in [11]). The locally  $\mathbb{Q}_p$ -rational representations in the title refer to the representations of the form  $\pi \otimes_L W$ , where  $\pi$  is a smooth

representation of  $G$  on an  $L$ -vector space and  $W$  is a  $\mathbb{Q}_p$ -rational representation as above. If  $\pi$  is absolutely irreducible then  $\pi \otimes W$  is also absolutely irreducible, (once one knows that the restriction of  $W$  to an open subgroup of  $G$  remains absolutely irreducible, the argument in [33] goes through).

**Lemma 7.1.** *Let  $\mathcal{G}$  be a group and  $U, V, W$  representations of  $\mathcal{G}$  on  $L$ -vector spaces then*

$$\mathrm{Hom}_{\mathcal{G}}(U \otimes_L V, W) \cong \mathrm{Hom}_{\mathcal{G}}(U, \mathrm{Hom}_L(V, W)),$$

where we consider  $\mathrm{Hom}_L(V, W)$  as a representation of  $\mathcal{G}$ , via  $[g \cdot \phi](v) = g(\phi(g^{-1}v))$ .

*Proof.* We have an isomorphism of  $L$ -vector spaces  $\mathrm{Hom}_L(U \otimes_L V, W) \cong \mathrm{Hom}_L(U, \mathrm{Hom}_L(V, W))$ ,  $\phi \mapsto \phi'$ , where  $[\phi'(u)](v) = \phi(u \otimes v)$ . Suppose that  $\phi$  is  $\mathcal{G}$ -equivariant then  $[\phi'(gu)](v) = \phi(gu \otimes v) = g\phi(u \otimes g^{-1}v) = [g \cdot \phi(u)](v)$ . Hence,  $\phi'$  is  $\mathcal{G}$ -equivariant. Suppose that  $\phi'$  is  $\mathcal{G}$ -equivariant then  $\phi(gu \otimes gv) = [\phi'(gu)](gv) = [g \cdot \phi'(u)](gv) = g\phi(u \otimes v)$ .  $\square$

**Theorem 7.2.** *Let  $(E, \|\cdot\|)$  be a unitary  $L$ -Banach space representation of  $G$ , such that  $\|E\| \subseteq |L|$ ,  $\varpi$  acts trivially and the restriction of  $E^0 \otimes_A k$  to  $K$  is an admissible injective object in  $\mathrm{Rep}_k(G)$ . Let  $\sigma$  be an irreducible  $k$ -representation of  $K$ , such that  $\mathrm{Hom}_K(\sigma, E^0 \otimes_A k) \neq 0$ . Let  $W$  be an irreducible  $\mathbb{Q}_p$ -rational representation of  $G$  tensored with a continuous character  $\eta : G \rightarrow L^\times$ . Let  $\tau$  be an absolutely irreducible smooth representation of  $K$ , such that  $\tau \otimes_L \overline{L}$  is typical for a Bernstein component  $\mathfrak{R}_s$ . Choose a  $K$ -invariant lattice  $M$  in  $\tau \otimes_L W$  and suppose that  $\sigma$  occurs as a subquotient of  $M \otimes_A k$ . Then there exists a finite extension  $L'$  of  $L$ , a smooth absolutely irreducible representation  $\pi$  on an  $L'$ -vector space, such that  $\pi_{\overline{L}}$  lies in  $\mathfrak{R}_s$ , and a  $G$ -equivariant embedding*

$$\pi \otimes_{L'} W_{L'} \hookrightarrow E_{L'}.$$

*Proof.* We note that any  $L$ -Banach space topology on a finite dimensional  $L$ -vector space  $V$  coincides with the finest locally convex one, [34, Prop. 4.13]. Hence any  $L$ -linear map from  $V$  into any locally convex  $L$ -vector space is continuous, [34, §5 C]. Hence, we have

$$\mathrm{Hom}_K(\tau \otimes_L W, E) \cong \mathrm{Hom}_{L[[K]]}(E^d, (\tau \otimes_L W)^d).$$

It follows from Proposition 5.6 and Corollary 4.9 that  $\mathrm{Hom}_K(\tau \otimes W, E)$  is finite dimensional. Moreover, since  $\sigma$  is a subquotient of  $M \otimes_A k$ , Proposition 5.6 and Corollary 4.9 imply that  $\mathrm{Hom}_K(\tau \otimes_L W, E)$  is non-zero. Since  $\varpi$  acts



by a scalar on  $W$ , there exists a unique extension of  $\tau$  to a representation  $\tilde{\tau}$  of  $KZ$  such that  $\varpi$  acts trivially on  $\tilde{\tau} \otimes_L W$ . Since  $\varpi$  act trivially on  $E$  we have

$$\begin{aligned} \mathrm{Hom}_K(\tau \otimes_L W, E) &\cong \mathrm{Hom}_{KZ}(\tilde{\tau} \otimes_L W, E) \cong \mathrm{Hom}_{KZ}(\tilde{\tau}, \mathrm{Hom}_L(W, E)) \\ &\cong \mathrm{Hom}_G(\mathrm{c}\text{-}\mathrm{Ind}_{KZ}^G \tilde{\tau}, \mathrm{Hom}_L(W, E)), \end{aligned} \quad (8)$$

where the second isomorphism is given by Lemma 7.1. Choose a non-zero  $\phi \in \mathrm{Hom}_G(\mathrm{c}\text{-}\mathrm{Ind}_{KZ}^G \tilde{\tau}, \mathrm{Hom}_L(W, E))$  and let  $\pi_1$  be the image of  $\phi$ . Since  $\pi_1$  is a smooth representation, it will be contained in  $\mathrm{Hom}_L^{sm}(W, E)$  consisting of  $\phi \in \mathrm{Hom}_L(W, E)$  such that there exists an open subgroup  $J$  of  $G$ , with  $g\phi g^{-1} = \phi$ , for all  $g \in J$ . Let  $J$  be an open compact subgroup of  $G$ , then the subspace of  $J$ -invariants in  $\mathrm{Hom}_L^{sm}(W, E)$  is equal to  $\mathrm{Hom}_J(W, E)$ , which is finite dimensional by Proposition 5.6 and Corollary 4.9. Hence,  $\mathrm{Hom}_L^{sm}(W, E)$  is an admissible representation of  $G$ . Since  $\pi_1$  is finitely generated [39, 5.10] implies that  $\pi_1$  is of finite length as  $L[G]$ -module. Let  $\pi_2$  be an irreducible  $L[G]$ -submodule of  $\pi_1$ . It follows from [39, 4.4] that there exists a finite extension  $L'$  of  $L$ , such that  $\pi_2 \otimes_L L'$  is a direct sum of absolutely irreducible representations. Let  $\pi$  be an irreducible summand. Since  $\mathrm{Hom}_L(W, E) \otimes_L L' \cong \mathrm{Hom}_{L'}(W_{L'}, E_{L'})$ , we get

$$\mathrm{Hom}_G(\pi \otimes_{L'} W_{L'}, E_{L'}) \cong \mathrm{Hom}_G(\pi, \mathrm{Hom}_{L'}(W_{L'}, E_{L'})) \neq 0.$$

Since  $\pi$  is irreducible and  $W$  is  $\mathbb{Q}_p$ -rational tensored with a character, the representation  $\pi \otimes_{L'} W_{L'}$  is irreducible, and hence any non-zero homomorphism is an injection.

It remains to show that  $\pi_{\overline{L}}$  lies in  $\mathfrak{R}_s$ . We know that  $\pi_{\overline{L}}$  is a subobject of  $\pi_1 \otimes_L \overline{L}$ , which is generated as a  $G$ -representation by a subspace isomorphic to  $\tau_{\overline{L}}$ . If  $\tau$  is a type, then we are done. If  $\tau$  is not a type, then  $\tau \cong \chi \circ \det$  or  $\tau \cong \mathrm{St} \otimes \chi \circ \det$ , for some smooth character  $\chi : \mathfrak{o}^\times \rightarrow L^\times$ , where  $\mathrm{St}$  denotes a lift to  $K$  of the Steinberg representation of  $K/K_1 \cong \mathrm{GL}_2(\mathbb{F}_q)$ . By twisting we may assume  $\chi$  to be trivial. Then the trivial representation of  $I$  is a type for  $\mathfrak{R}_s$ . Since  $\mathrm{Hom}_I(\mathbf{1}, \tau) \neq 0$ , we get that  $\pi_1 \otimes_L \overline{L}$  is generated by a subspace isomorphic to the trivial representation of  $I$ , and hence  $\pi_{\overline{L}}$  lies in  $\mathfrak{R}_s$ .  $\square$

We also give a variant of Theorem 7.2 when  $\tau$  is the trivial representation of  $K$ .

**Lemma 7.3.** *Let  $V$  be an absolutely irreducible representation of  $G$  on a finite dimensional  $L$ -vector space. Suppose that  $G$  stabilizes a lattice in  $V$  then  $V$  is a character.*

*Proof.* If  $G$  stabilizes a lattice in  $V$ , then we obtain a group homomorphism  $\rho : G \rightarrow \mathrm{GL}_d(A)$ , where  $d = \dim_L V$ . It follows from [26, Thm. 8.4] that  $\mathrm{SL}_2(F)$  does not have a non-trivial quotient of finite order. Hence,  $\rho(\mathrm{SL}_2(F))$  is contained in  $1 + \varpi_L^n M_d(A)$ , for all  $n \geq 1$ . Since the intersection of these groups is trivial,  $\rho(\mathrm{SL}_2(F)) = 1$ . Hence,  $\rho(G)$  is abelian and so for  $g \in G$ , the map  $v \mapsto gv$  lies in  $\mathrm{End}_G(V)$ . Since  $V$  is finite dimensional and absolutely irreducible, Schur's lemma gives  $\mathrm{End}_G(V) = L$ , and hence  $G$  acts by a character.  $\square$

**Corollary 7.4.** *Let  $E, \sigma, W$  be as in Theorem 7.2. Let  $M$  be a  $K$ -stable lattice in  $W$  and suppose that  $\sigma$  is a subquotient of  $M \otimes_A k$ . Assume that  $W$  is not one dimensional then there exists a finite extension  $L'$  of  $L$  and an unramified principal series representation  $\pi$  of  $G$  on an  $L'$ -vector space and a  $G$ -equivariant injection  $\pi \otimes_{L'} W_{L'} \hookrightarrow E_{L'}$ .*

*Proof.* We proceed as in the proof of Theorem 7.2 with  $\tau = \mathbf{1}$ . Let  $\tilde{\mathbf{1}}$  denote an unramified character such that  $\varpi$  acts trivially on  $\tilde{\mathbf{1}} \otimes W$ . The equation (8) implies that  $N := \mathrm{Hom}_G(\mathrm{c}\text{-Ind}_{KZ}^G \tilde{\mathbf{1}}, \mathrm{Hom}_L(W, E))$  is a non-zero finite dimensional vector space. It is also naturally a module for the Hecke algebra  $\mathcal{H} := \mathrm{End}_G(\mathrm{c}\text{-Ind}_{KZ}^G \tilde{\mathbf{1}})$ . The Hecke algebra  $\mathcal{H}$  is isomorphic to  $L[T]$  the polynomial ring in one variable. So if we choose some non-zero  $\phi \in N$  then the  $\mathcal{H}$ -module generated by  $\phi$  is isomorphic to  $L[T]/(P)$ , for some polynomial  $P$ . Hence if we let  $\pi_1$  be the image of  $\phi$ , then  $\pi_1$  is isomorphic to

$$\mathrm{c}\text{-Ind}_{KZ}^G \tilde{\mathbf{1}} \otimes_{L[T]} L[T]/(P) \cong \mathrm{c}\text{-Ind}_{KZ}^G \tilde{\mathbf{1}}/(P).$$

Let  $L'$  be the splitting field of  $P$ , and  $a$  a root of  $P$ . If we set  $\pi := \frac{\mathrm{c}\text{-Ind}_{KZ}^G \tilde{\mathbf{1}}_{L'}}{(T-a)}$ , then  $\pi$  will be isomorphic to a subobject of  $\pi_1 \otimes_L L'$ , so we have

$$\mathrm{Hom}_G(\pi \otimes_{L'} W_{L'}, E_{L'}) \cong \mathrm{Hom}_G(\pi, \mathrm{Hom}_{L'}(W_{L'}, E_{L'})) \neq 0.$$

Now  $\pi$  is an unramified principal series representation. If  $\pi$  is irreducible we are done. If  $\pi$  is reducible then it is a non-split extension  $0 \rightarrow \mathrm{St} \otimes \chi \rightarrow \pi \rightarrow \chi \rightarrow 0$ , for some character  $\chi : G \rightarrow L^\times$ , and  $\mathrm{St}$  denotes the Steinberg representation of  $G$ . The sequence  $0 \rightarrow \mathrm{St} \otimes W_\chi \rightarrow \pi \otimes W \rightarrow W_\chi \rightarrow 0$  is also non-split, otherwise by tensoring with  $W^*$  and taking smooth vectors we would obtain the splitting of the original sequence. So if there exists non-zero  $\psi \in \mathrm{Hom}_G(\pi \otimes_{L'} W_{L'}, E_{L'})$ , such that  $\mathrm{Ker} \psi \neq 0$  then the image of  $\psi$  is isomorphic to  $W_\chi$ . But then  $W_\chi \cap E^0$  would be a  $G$ -invariant lattice in  $W_\chi$ , which would contradict Lemma 7.3.  $\square$

Since all the  $L$ -Banach spaces below arise from the constructions of Theorem 7.2 and Corollary 7.4 we always assume that  $\|E\| \subseteq |L|$ .

**Corollary 7.5.** *Let  $\tau$  be a smooth absolutely irreducible  $L$ -representation of  $K$ , which is typical for Bernstein component  $\mathfrak{R}_s$ . Let  $W$  be a  $\mathbb{Q}_p$ -rational representation of  $G$ , twisted by a continuous character. Let  $M$  be a  $K$ -invariant lattice in  $\tau \otimes W$ . Let  $\kappa$  be an absolutely irreducible smooth admissible  $k$ -representation of  $G$ , such that  $\varpi$  acts trivially. Suppose that there exists an irreducible  $k$ -representation  $\sigma$  of  $K$ , such that*

- (1)  $\text{Hom}_K(\sigma, \kappa) \neq 0$ ;
- (2)  $\sigma$  occurs as a subquotient of  $M \otimes_A k$ .

*Then there exists a finite extension  $L'$  of  $L$ , an absolutely irreducible smooth  $L'$ -representation  $\pi$  of  $G$  in  $\mathfrak{R}_s$ , and an admissible unitary  $L'$ -Banach space representation  $(E, \|\cdot\|)$  of  $G$ , such that the following hold:*

- (i)  $\pi \otimes_{L'} W_{L'}$  is a dense  $G$ -invariant subspace of  $E$ ;
- (ii)  $\text{Hom}_G(\kappa_{k'}, E^0 \otimes_{A'} k') \neq 0$ .

*Proof.* Since  $\kappa$  is admissible, by [12, Cor.9.11] there exists a  $G$ -equivariant embedding  $\kappa \hookrightarrow \Omega$ , where  $\Omega$  is a smooth representation of  $G$ , such that  $\Omega|_K$  is an injective envelope of  $\text{soc}_K \kappa$  in  $\text{Rep}_k(K)$ . Let  $(E', \|\cdot\|)$  be a lift of  $\Omega$  as in Theorem 6.1. Then by Theorem 7.2 there exists a finite extension  $L'$  of  $L$  and an absolutely irreducible smooth  $L'$ -representation  $\pi$  in  $\mathfrak{R}_s$ , and a  $G$ -equivariant embedding  $\pi \otimes_{L'} W_{L'} \hookrightarrow E'_{L'}$ . Let  $E$  be the closure of  $\pi \otimes_{L'} W_{L'}$  in  $E'$  and  $E^0$  the unit ball in  $E$  with respect to  $\|\cdot\|$ . Since  $E'$  is admissible, so is  $E$ . Lemma 5.5 gives a  $G$ -equivariant injection  $E^0 \otimes_{A'} k' \hookrightarrow \Omega_{k'}$ . It follows from Lemma 3.9 that  $\text{soc}_K \Omega_{k'} \cong (\text{soc}_K \Omega) \otimes_k k' \cong (\text{soc}_K \kappa) \otimes_k k'$ , hence  $\kappa_{k'} \cap (E^0 \otimes_{A'} k') \neq 0$ . Since  $\kappa$  is absolutely irreducible we get that  $\kappa_{k'}$  is contained in  $E^0 \otimes_{A'} k'$ .  $\square$

**Corollary 7.6.** *Let  $W$  be  $\mathbb{Q}_p$ -rational representation of  $G$ , twisted by a continuous character. Let  $M$  be a  $K$ -invariant lattice in  $W$ . Let  $\kappa$  be an absolutely irreducible smooth admissible  $k$ -representation of  $G$ , such that  $\varpi$  acts trivially. Suppose that there exists an irreducible  $k$ -representation  $\sigma$  of  $K$ , such that*

- (1)  $\text{Hom}_K(\sigma, \kappa) \neq 0$ ;
- (2)  $\sigma$  occurs as a subquotient of  $M \otimes_A k$ .

Assume that either  $\kappa$  is not finite dimensional or  $W$  is not a character. Then there exists a finite extension  $L'$  of  $L$ , an unramified smooth principal series  $L'$ -representation  $\pi$  of  $G$ , and an admissible unitary  $L'$ -Banach space representation  $(E, \|\cdot\|)$  of  $G$ , such that the following hold:

- (i)  $\pi \otimes_{L'} W_{L'}$  is a dense  $G$ -equivariant subspace of  $E$ ;
- (ii)  $\text{Hom}_G(\kappa_{k'}, E^0 \otimes_{A'} k') \neq 0$ .

*Proof.* The proof is the same as of Corollary 7.5, using Corollary 7.4 instead of Theorem 7.2.  $\square$

**Remark 7.7.** We note that any irreducible smooth  $k$ -representation  $\sigma$  of  $K$  is of the form

$$\bigotimes_{\tau: k_F \hookrightarrow k} (\text{Sym}^{r_\tau} k^2 \otimes \det^{a_\tau})^\tau$$

with  $0 \leq r_\tau \leq p-1$  and  $0 \leq a_\tau < p-1$ . Hence, we may lift  $\sigma$  to a  $\mathbb{Q}_p$ -rational representation

$$W := \bigotimes_{\tilde{\tau}: F \hookrightarrow L} (\text{Sym}^{r_\tau} L^2 \otimes \det^{a_\tau})^{\tilde{\tau}},$$

where for each  $\tau$  we fix  $\tilde{\tau}: F \hookrightarrow L$ , inducing  $\tau$  on the residue fields.

We also note that although we know that the completion  $E$  in Corollaries 7.5, 7.6 is admissible, we do not know in general whether it is of finite length as a (topological) representation of  $G$ .

**Lemma 7.8.** Let  $\pi$  be a smooth  $L$ -representation of  $G$ ;  $\tau$  a smooth irreducible representation of  $K$ , such that  $\text{Hom}_K(\tau, \pi) \neq 0$ ;  $W$  a  $\mathbb{Q}_p$ -rational representation of  $G$ , twisted by a continuous character. Suppose that  $(E, \|\cdot\|)$  is a unitary  $L$ -Banach space representation of  $G$ , which contains  $\pi \otimes_L W$  as a dense  $G$ -invariant subspace. Choose a  $K$ -invariant lattice  $M$  in  $\tau \otimes_L W$ , then there exists an irreducible subquotient  $\sigma$  of  $M \otimes_A k$  such that  $\text{Hom}_K(\sigma, E^0 \otimes_A k) \neq 0$ .

*Proof.* Let  $M' := (\tau \otimes_L W) \cap E^0$ , then Lemma 5.5 gives an injection  $M' \otimes_A k \hookrightarrow E^0 \otimes_A k$ . In particular, there exists some irreducible  $k$ -representation  $\sigma$  of  $K$ , such that  $\text{Hom}_K(\sigma, M' \otimes_A k) \neq 0$  and  $\text{Hom}_K(\sigma, E^0 \otimes_A k) \neq 0$ . Since  $\tau \otimes_L W$  is finite dimensional we have  $(M' \otimes_A k)^{ss} = (M \otimes_A k)^{ss}$ .  $\square$

Lemma 7.8 together with Proposition 6.2 shows that any admissible completion of  $\pi \otimes W$  arises from our construction.

**Lemma 7.9.** *Let  $\chi_1, \chi_2 : F^\times \rightarrow L^\times$  be smooth characters and let  $\pi := \text{Ind}_B^G \chi_1 \otimes \chi_2 | \cdot |^{-1}$  be a smooth  $L$ -representation of  $G$ . For each  $\sigma \in \Sigma$ , let  $r_\sigma \geq 0$  be an integer, set  $W := \otimes_{\sigma \in \Sigma} (\text{Sym}^{r_\sigma} L^2)^\sigma$ , and let  $\eta : F^\times \rightarrow L^\times$  be a continuous character. Suppose that  $\pi \otimes W \otimes \eta \circ \det$  admits a unitary completion and set  $\lambda_1 := \chi_1(\varpi)^{-1}$  and  $\lambda_2 := \chi_2(\varpi)^{-1}$ , then the following hold:*

$$(i) \quad (\text{val}(\lambda_1) - \text{val}(\eta(\varpi)) + (\text{val}(\lambda_2) - \text{val}(\eta(\varpi)))) = (1 + \sum_{\sigma \in \Sigma} r_\sigma)/e;$$

$$(ii) \quad \text{val}(\lambda_2) - \text{val}(\eta(\varpi)) \geq 0, \text{val}(\lambda_1) - \text{val}(\eta(\varpi)) \geq 0;$$

where  $e := e(F|\mathbb{Q}_p)$  is the ramification index. Moreover, if  $\varpi \in Z$  acts trivially on  $\pi \otimes W \otimes \eta \circ \det$  then

$$\lambda_1 \lambda_2 = p^{1/e} \eta(\varpi)^2 \prod_{\sigma \in \Sigma} \sigma(\varpi)^{r_\sigma}.$$

*Proof.* We have  $\pi \otimes W \otimes \eta \circ \det \cong (\pi \otimes |\eta|^{-1} \circ \det) \otimes W \otimes (\eta|\eta|) \circ \det$ . The character  $\eta|\eta|$  is unitary, and if we let  $\chi'_1 := \chi_1|\eta|^{-1}$  and  $\chi'_2 := \chi_2|\eta|^{-1}$  then the characters  $\chi'_1$  and  $\chi'_2$  are smooth and  $\pi \otimes |\eta|^{-1} \circ \det \cong \text{Ind}_B^G \chi'_1 \otimes \chi'_2 | \cdot |^{-1}$ . Since  $\text{val}(\chi'_i(\varpi)) = \text{val}(\chi_i(\varpi)) + \text{val}(\eta(\varpi))$ , for  $i \in \{1, 2\}$ , we may assume that  $\eta$  is the trivial character.

Note that  $\varpi$  acts on  $\pi \otimes_L W$  by a scalar  $\lambda_1^{-1} p^{1/e} \lambda_2^{-1} \prod_{\sigma \in \Sigma} \sigma(\varpi)^{r_\sigma}$ . If  $\pi \otimes W$  admits a unitary completion then the central character of  $\pi \otimes W$  has to be unitary, which is equivalent to  $\varpi$  acting by a scalar in  $A^\times$ . This implies

$$\text{val}(\lambda_1) + \text{val}(\lambda_2) = (1 + \sum_{\sigma \in \Sigma} r_\sigma)/e.$$

This gives the first and also the last assertion.

Let  $\varphi \in \pi$  be the function such that  $\text{Supp } \varphi = BsI = Bs(I_1 \cap U)$  and  $\varphi(su) = 1$ , for all  $u \in I_1 \cap U$ . Then  $t\varphi$  is the unique function in  $\pi$  with the support  $(\text{Supp } \varphi)t^{-1} = Bs(K_1 \cap U)$  and  $[t\varphi](su) = \chi_2(\varpi)|\varpi|^{-1}$ , for all  $u \in K_1 \cap U$ . Hence, we have

$$\sum_{\lambda \in \mathfrak{o}/\mathfrak{p}} \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} t\varphi = \lambda_2^{-1} p^{1/e} \varphi \quad (9)$$

Let  $X^{\mathbf{r}} \in W$ ,  $X^{\mathbf{r}} = \otimes_{\sigma \in \Sigma} X^{r_\sigma}$ . Then  $U$  acts trivially on  $X^{\mathbf{r}}$  and  $tX^{\mathbf{r}} = (\prod_{\sigma \in \Sigma} \sigma(\varpi)^{r_\sigma}) X^{\mathbf{r}}$ . Hence, (9) gives us

$$\sum_{\lambda \in \mathfrak{o}/\mathfrak{p}} \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} t(\varphi \otimes X^{\mathbf{r}}) = (\lambda_2^{-1} p^{1/e} \prod_{\sigma \in \Sigma} \sigma(\varpi)^{r_\sigma}) \varphi \otimes X^{\mathbf{r}}. \quad (10)$$

Suppose that  $\pi \otimes W$  admits a unitary completion. Since the action of  $G$  is unitary, the triangle inequality applied to (10) gives  $\lambda_2^{-1} p^{1/e} \prod_{\sigma \in \Sigma} \sigma(\varpi)^{r_\sigma} \in A$ . Hence,  $\text{val}(\lambda_2) \leq (1 + \sum_{\sigma \in \Sigma} r_\sigma)/e$ . If  $\chi_1 = \chi_2| \cdot |$  then  $\text{val}(\lambda_1) < \text{val}(\lambda_2)$  and so (ii) follows from (i). If  $\chi_1 = \chi_2| \cdot |^{-1}$  then  $W \otimes \chi_1 \circ \det$  is a subrepresentation of  $\pi \otimes W$ . Hence, we obtain a  $G$ -invariant norm on  $W \otimes \chi_1 \circ \det$ . Lemma 7.3 implies that  $r_\sigma = 0$  for all  $\sigma \in \Sigma$ . Part (i) gives  $\text{val}(\lambda_1) + \text{val}(\lambda_2) = 1/e$  and so  $\text{val}(\lambda_2) = 1/e$  and  $\text{val}(\lambda_1) = 0$ .

Suppose that  $\chi_1 \neq \chi_2| \cdot |^{\pm 1}$  then  $\pi$  is irreducible and the intertwining operator induces an isomorphism  $\pi \cong \text{Ind}_B^G \chi_2 \otimes \chi_1| \cdot |^{-1}$ , see [14, Thm 4.5.3]. Hence, we also obtain  $\text{val}(\lambda_1) \leq (1 + \sum_{\sigma \in \Sigma} r_\sigma)/e$ , which implies (ii).  $\square$

Let  $\chi_1, \chi_2 : F^\times \rightarrow L^\times$  be smooth characters and let  $\pi := \text{Ind}_B^G \chi_1 \otimes \chi_2| \cdot |^{-1}$  be a smooth  $L$ -representation of  $G$ . Set  $\lambda_1 := \chi_1(\varpi)^{-1}$  and  $\lambda_2 = \chi_2(\varpi)^{-1}$ . For each  $\sigma \in \Sigma$ , let  $r_\sigma \geq 0$  be an integer, set  $W := \otimes_{\sigma \in \Sigma} (\text{Sym}^{r_\sigma} L^2)^\sigma$ , and let  $\eta : F^\times \rightarrow L^\times$  be a continuous character. Suppose that  $\pi \otimes W \otimes \eta \circ \det$  is a dense  $G$ -invariant subspace in a unitary  $L$ -Banach space representation  $E$  of  $G$ . Let  $\theta : F^\times \rightarrow L^\times$  be the character  $\theta(x) := \prod_{\sigma \in \Sigma} \sigma(x)^{r_\sigma}$ . Assume that  $\text{val}(\lambda_1) = \text{val}(\eta(\varpi))$ , then it follows from Lemma 7.9 that the characters  $\chi_1 \eta$  and  $\chi_2| \cdot |^{-1} \eta \theta$  are integral. Let

$$\psi_1 := \chi_1 \eta \pmod{1 + \mathfrak{M}}, \quad \psi_2 := \chi_2| \cdot |^{-1} \eta \theta \pmod{1 + \mathfrak{M}}.$$

**Lemma 7.10.** *Assume that we are in the situation as above. Let  $\| \cdot \|$  be a  $G$ -invariant norm defining the topology on  $E$  and let  $E^0$  be the unit ball with respect to  $\| \cdot \|$  then  $\text{Hom}_G(\text{Ind}_B^G \psi_1 \otimes \psi_2, E^0 \otimes_A k) \neq 0$ .*

*Proof.* We note that the assertion is true for  $\pi$  if and only if it is true for  $\pi \otimes \xi \circ \det$ , for some  $\xi : F^\times \rightarrow A^\times$  an unramified character. Hence, we may assume that  $\eta$  is trivial, and  $\varpi$  acts trivially (possibly after replacing  $L$  with a quadratic extension). Let  $\phi \in \pi$  and  $X^{\mathbf{r}} \in W$  be as in the proof of Lemma 7.9. We may assume that  $\|\phi \otimes X^{\mathbf{r}}\| = 1$ . Let  $v$  be the image of  $\phi \otimes X^{\mathbf{r}}$  in  $E^0 \otimes_A k$ , then  $v \neq 0$ , and  $I_1 \cap B$  acts trivially on  $v$ ,

$$\begin{pmatrix} [\lambda] & 0 \\ 0 & [\mu] \end{pmatrix} v = \psi_2([\lambda]) \psi_1([\mu]) v, \quad \forall \lambda, \mu \in k_F.$$

Set  $u := \lambda_2^{-1} p^{1/e} \prod_{\sigma \in \Sigma} \sigma(\varpi)^{r_\sigma}$ , then Lemma 7.9 implies that  $u$  is a unit in  $A$ . Let  $\bar{u}$  be the image of  $u$  in  $k$ . Then (10) reduces to:

$$\sum_{\lambda \in \mathfrak{o}/\mathfrak{p}} \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} t v = \bar{u} v. \tag{11}$$

Since  $\bar{u} \neq 0$  we are in the situation described in the proof of [31, Thm. 5.4], ( $v$  corresponds to  $\phi_2$ ). The argument there gives:

- (i)  $v$  is fixed by  $I_1$ ;
- (ii)  $\sigma := \langle K \cdot v \rangle$  is an irreducible representation of  $K$ , and if  $\psi_1|_{\mathfrak{o}^\times} = \psi_2|_{\mathfrak{o}^\times}$  then  $\sigma \cong \text{St} \otimes \psi_1 \circ \det$ ;

Now (11) and [31, Lem 3.1] together with [1, Thm. 30] implies that the map  $\text{c-Ind}_{KZ}^G \sigma \rightarrow \langle G \cdot v \rangle$  factors through  $\text{Ind}_B^G \xi_1 \otimes \xi_2$ , where  $\xi_1|_{\mathfrak{o}^\times} = \psi_1$ ,  $\xi_2|_{\mathfrak{o}^\times} = \psi_2$ ,  $\xi_2(\varpi) = \xi_1(\varpi^{-1}) = \bar{u}$ . This gives the result.  $\square$

One may call the situation of Lemma 7.10 the ordinary case. Lemma 7.10 shows that most of the time the completions we obtain in Corollaries 7.5 and 7.6 are not ordinary.

## 8 The case $F = \mathbb{Q}_p$

We assume that  $F = \mathbb{Q}_p$  and  $p > 2$ , and we study in more detail the consequences of Corollaries 7.5, 7.6. Barthel-Livné have shown in [1] that smooth irreducible  $\bar{k}$ -representations of  $G$ , with the central character fall into four disjoint classes:

- (1)  $\chi \circ \det$ ;
- (2)  $\text{Sp} \otimes \chi \circ \det$ ;
- (3)  $\text{Ind}_B^G \chi_1 \otimes \chi_2$ ,  $\chi_1 \neq \chi_2$ ;
- (4) supersingular;

where  $\text{Sp}$  is the Steinberg representation defined by the exact sequence  $0 \rightarrow \mathbf{1} \rightarrow \text{Ind}_B^G \mathbf{1} \rightarrow \text{Sp} \rightarrow 0$ . Breuil in [9] has classified the supersingular representations. We recall the classification. Fix an integer  $0 \leq r \leq p-1$ , then the representation  $\text{Sym}^r \bar{k}^2$  of  $K$  is irreducible. We put the action of  $KZ$  on  $\text{Sym}^r \bar{k}^2$  by making  $p$  act trivially. Let  $\mathcal{H}$  be the Hecke algebra,  $\mathcal{H} := \text{End}_G(\text{c-Ind}_{KZ}^G \text{Sym}^r \bar{k}^2)$ . Proposition 8 of [1] asserts that as a  $\bar{k}$ -algebra  $\mathcal{H}$  is isomorphic to a polynomial ring in one variable  $\bar{k}[T]$ , where  $T \in \mathcal{H}$  is an endomorphism defined in [1, §3]. Moreover,  $\text{c-Ind}_{KZ}^G \text{Sym}^r \bar{k}^2$  is a free  $\mathcal{H}$ -module, [1, Thm.19]. Define,

$$\kappa(r) := \frac{\text{c-Ind}_{KZ}^G \text{Sym}^r \bar{k}^2}{(T)},$$

and if  $\eta : \mathbb{Q}_p^\times \rightarrow \bar{k}^\times$  is a smooth character, then set  $\kappa(r, \eta) := \kappa(r) \otimes \eta \circ \det$ . Breuil has shown [9, Thm.1.1] that the representations  $\kappa(r, \eta)$  are irreducible and any irreducible supersingular representation of  $G$  is isomorphic to  $\kappa(r, \eta)$ , for some  $0 \leq r \leq p-1$  and  $\eta$ . All the isomorphism between supersingular representations corresponding to different  $r$  and  $\eta$  are given by

$$\kappa(r, \eta) \cong \kappa(r, \eta\mu_{-1}) \cong \kappa(p-1-r, \eta\omega^r) \cong \kappa(p-1-r, \eta\omega^r\mu_{-1}) \quad (12)$$

see [9, Thm. 1.3], where  $\omega : \mathbb{Q}_p^\times \rightarrow \bar{k}^\times$  is a character given by  $\omega(p) = 1$  and  $\omega|_{\mathbb{Z}_p^\times}$  is the natural map  $\mathbb{Z}_p^\times \rightarrow \mathbb{F}_p^\times \rightarrow \bar{k}^\times$ , and given  $\lambda \in \bar{k}$ , we denote by  $\mu_\lambda : \mathbb{Q}_p^\times \rightarrow \bar{k}^\times$  the unramified character  $x \mapsto \lambda^{\text{val}(x)}$ .

**Lemma 8.1.** *Every smooth irreducible  $\bar{k}$ -representation of  $G$  with a central character can be realized over a finite extension of  $k$ .*

*Proof.* Since  $\mathbb{Q}_p^\times$  is finitely generated as a topological group, given a smooth character  $\eta : \mathbb{Q}_p^\times \rightarrow \bar{k}^\times$ , the image  $\eta(\mathbb{Q}_p^\times)$  lies in a finite extension of  $k$ . Hence, all the principal series representations in (3) can be realized over a finite extension and if  $\kappa$  can be realized over a finite extension, then so can  $\kappa \otimes \eta \circ \det$ . It is clear that the trivial representation can be realized over  $\mathbb{F}_p$ , and then we may realize  $\text{Sp}$  as the quotient  $0 \rightarrow \mathbb{F}_p \rightarrow \text{Ind}_B^G \mathbb{F}_p \rightarrow \text{Sp} \rightarrow 0$ . Moreover, we may realize  $\kappa(r)$  over  $\mathbb{F}_p$  as  $\frac{\text{c-Ind}_{KZ}^G \text{Sym}^r \mathbb{F}_p^2}{(T)}$ , as  $T$  in this case is defined over  $\mathbb{F}_p$ , [1, Prop 8].  $\square$

**Lemma 8.2.** *Let  $\sigma := \text{Sym}^r \bar{k}^2 \otimes \det^a$ , with  $0 \leq r \leq p-1$ ,  $0 \leq a < p-1$ . Let  $\kappa$  be an irreducible smooth  $\bar{k}$ -representation of  $G$ , such that  $p \in Z$  acts trivially on  $\kappa$ . Then  $\text{Hom}_K(\sigma, \kappa) \neq 0$  if and only if one of the following holds:*

- (i)  $r = 0$  and  $\kappa$  is isomorphic to one of the following:  $\kappa(0, \omega^a)$ ,  $(\mu_{\pm 1} \omega^a) \circ \det$  or  $\text{Ind}_B^G \mu_{\lambda^{-1}} \omega^a \otimes \mu_\lambda \omega^a$ , for  $\lambda \in \bar{k}^\times \setminus \{\pm 1\}$ ;
- (ii)  $r = p-1$  and  $\kappa$  is isomorphic to one of the following:  $\kappa(p-1, \omega^a)$ ,  $\text{Sp} \otimes (\mu_{\pm 1} \omega^a) \circ \det$  or  $\text{Ind}_B^G \mu_{\lambda^{-1}} \omega^a \otimes \mu_\lambda \omega^a$ , for  $\lambda \in \bar{k}^\times \setminus \{\pm 1\}$ ;
- (iii)  $0 < r < p-1$  and  $\kappa$  is isomorphic to one of the following:  $\kappa(r, \omega^a)$  or  $\kappa \cong \text{Ind}_B^G \mu_{\lambda^{-1}} \omega^a \otimes \mu_\lambda \omega^{a+r}$ , for  $\lambda \in \bar{k}^\times$ ;

*Proof.* This is well known.  $\square$

**Remark 8.3.** *Since  $k_L$  contains  $\mathbb{F}_p$  every irreducible  $k_L$ -representation of  $K$  is absolutely irreducible. Lemma 3.9 allows us to use Lemma 8.2 with  $k_L$  instead of  $\bar{k}$ .*



**Lemma 8.4.** *Let  $\psi_1, \psi_2 : \mathbb{Q}_p^\times \rightarrow k_L^\times$  be smooth characters. Suppose that  $\text{Ind}_B^G \psi \otimes \psi_2$  has an irreducible subquotient  $\kappa$  with  $\text{soc}_K \kappa \cong \text{Sym}^r k_L^2 \otimes \det^a$ , with  $0 \leq r \leq p-1$ ,  $0 \leq a < p-1$ , then  $(\psi_1|_{\mathbb{Z}_p^\times}, \psi_2|_{\mathbb{Z}_p^\times}) = (\omega^a, \omega^{a+r})$ .*

*Proof.* If  $\text{Ind}_B^G \psi_1 \otimes \psi_2$  is irreducible then the assertion follows from Lemma 8.2 and [1, Thm 34 (2)]. If  $\text{Ind}_B^G \psi_1 \otimes \psi_2$  is reducible, then  $\psi_1 = \psi_2$ , and  $(\text{Ind}_B^G \psi_1 \otimes \psi_2)^{ss} \cong \psi_1 \circ \det \oplus \text{Sp} \otimes \psi_1 \circ \det$ , [1, Thm 30 (1)]. Hence,  $r = p-1$  or  $r = 0$  and so  $(\psi_1|_{\mathbb{Z}_p^\times}, \psi_2|_{\mathbb{Z}_p^\times}) = (\omega^a, \omega^a)$ .  $\square$

Let  $\theta_1, \theta_2 : \mathbb{Z}_p^\times \rightarrow L^\times$  be smooth characters. If  $\theta_1 = \theta_2$  we set  $c(\theta_1, \theta_2) := 0$ ,  $J_c := K$  and  $\tau(\theta_1, \theta_2) := \theta_1 \circ \det$ . If  $\theta_1 \neq \theta_2$  let  $c(\theta_1, \theta_2) \geq 1$  be the smallest integer  $c$ , such that  $\theta_1 \theta_2^{-1}$  is trivial on  $1 + p^c \mathbb{Z}_p$ . We set

$$J_c := \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^c \mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix},$$

and we consider  $\theta_1 \otimes \theta_2$  as a character of  $J_c$ , by

$$\theta_1 \otimes \theta_2 \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) := \theta_1(a) \theta_2(d).$$

Set  $\tau(\theta_1, \theta_2) := \text{Ind}_{J_c}^K \theta_1 \otimes \theta_2$ . If  $\theta_1 \neq \theta_2$  then  $\tau(\theta_1, \theta_2)$  is a type for the Bernstein component containing representations  $\text{Ind}_B^G \chi_1 \otimes \chi_2$  with  $\chi_1|_{\mathbb{Z}_p^\times} = \theta_1$  and  $\chi_2|_{\mathbb{Z}_p^\times} = \theta_2$ , see [25, A.2.2]. If  $\theta_1 = \theta_2$  then  $\tau(\theta_1, \theta_2)$  is typical for the Bernstein component containing  $\chi \circ \det$ , with  $\chi|_{\mathbb{Z}_p^\times} = \theta_1 = \theta_2$ .

**Theorem 8.5.** *Let  $\sigma := \text{Sym}^r k_L^2 \otimes \det^a$  with  $0 \leq r \leq p-1$  and  $0 \leq a < p-1$ . And let  $\kappa$  be an absolutely irreducible smooth  $k_L$ -representation with a central character, such that  $\text{Hom}_K(\sigma, \kappa) \neq 0$  and  $p \in Z$  acts trivially on  $\kappa$ . Fix an integer  $k \geq 2$  and smooth characters  $\theta_1, \theta_2 : \mathbb{Z}_p^\times \rightarrow L^\times$ . Let  $M$  be a  $K$ -stable lattice in  $\tau(\theta_1, \theta_2) \otimes \text{Sym}^{k-2} L^2$ . Suppose that  $\sigma$  is a subquotient of  $(M \otimes_A k_L)^{ss}$  then there exists a finite extension  $L'$  of  $L$ , smooth characters  $\chi_1, \chi_2 : \mathbb{Q}_p^\times \rightarrow (L')^\times$  and an admissible unitary completion  $E$  of  $(\text{Ind}_B^G \chi_1 \otimes \chi_2|_{\bullet} |^{-1}) \otimes \text{Sym}^{k-2} (L')^2$  such that the following hold:*

- (1)  $\chi_1 \neq \chi_2$ ;
- (2)  $\chi_1|_{\mathbb{Z}_p^\times} = \theta_1, \chi_2|_{\mathbb{Z}_p^\times} = \theta_2$ ;
- (3)  $\chi_1(p) \chi_2(p) p^{k-1} = 1$ ;
- (4)  $\text{val}(\chi_1(p)) \leq 0, \text{val}(\chi_2(p)) \leq 0$ ;

(5)  $\text{Hom}_G(\kappa, E^0 \otimes_{A'} k') \neq 0$ .

Moreover, if we assume that  $\kappa$  is not a subquotient of any principal series representation  $\text{Ind}_B^G \psi_1 \otimes \psi_2$ , such that  $(\psi_1|_{\mathbb{Z}_p^\times}, \psi_2|_{\mathbb{Z}_p^\times}) = (\bar{\theta}_1, \bar{\theta}_2 \omega^{k-2})$  or  $(\psi_1|_{\mathbb{Z}_p^\times}, \psi_2|_{\mathbb{Z}_p^\times}) = (\bar{\theta}_2, \bar{\theta}_1 \omega^{k-2})$ , then  $\text{val}(\chi_1(p)) < 0$  and  $\text{val}(\chi_2(p)) < 0$ .

*Proof.* Corollaries 7.5 and 7.6 give us an extension  $L'$  of  $L$ , an  $L'$ -principal series representation  $\pi := \text{Ind}_B^G \chi_1 \otimes \chi_2 | \cdot |^{-1}$ , such that  $\chi_1, \chi_2$  satisfy (2), (3) and an admissible unitary completion  $E$  of  $\pi \otimes \text{Sym}^{k-2}(L')^2$ , satisfying (5). For simplicity we assume that  $L = L'$ . Assume that  $\chi_1 = \chi_2$ , then [32, Cor 4.5] says that there exists  $x \in 1 + \mathfrak{M}$ ,  $x^2 \neq 1$  and an admissible unitary completion  $E_x$  of

$$(\text{Ind}_B^G \chi_1 \delta_x \otimes \chi_2 \delta_{x^{-1}} | \cdot |^{-1}) \otimes \text{Sym}^{k-2} L^2,$$

such that  $E_x^0 \otimes_A k_L \cong E^0 \otimes_A k_L$  as  $G$ -representations, where  $\delta_x : \mathbb{Q}_p^\times \rightarrow L^\times$  is an unramified character with  $\delta_x(p) = x$ . Hence, we may assume that  $\chi_1 \neq \chi_2$ . The condition (4) follows from Lemma 7.9. The last part follows from Lemma 7.10.  $\square$

We use the results of Berger-Breuil [4] and Berger [5] to transfer the statement of Theorem 8.5 to the Galois side. Let  $\mathcal{G}_{\mathbb{Q}_p}$  be the absolute Galois group of  $\mathbb{Q}_p$ , and let  $\mathcal{I}_{\mathbb{Q}_p}$  be the inertia subgroup. We consider characters of  $\mathbb{Q}_p^\times$  as characters of  $\mathcal{G}_{\mathbb{Q}_p}$  via class field theory, sending the geometric Frobenius to  $p$ , with this identification  $\omega$  is the reduction modulo  $p$  of the cyclotomic character. By a 2-dimensional  $L$ -linear representation of  $\mathcal{G}_{\mathbb{Q}_p}$  we mean a continuous group homomorphism  $\mathcal{G}_{\mathbb{Q}_p} \rightarrow \text{GL}_2(L)$ , where  $\text{GL}_2(L)$  is equipped with the  $p$ -adic topology inherited from  $L$ . Since  $\mathcal{G}_{\mathbb{Q}_p}$  is compact, it will stabilize some  $A$ -lattice  $T$  in  $V$ . Now  $(T \otimes_A k_L)^{ss}$  does not depend on the choice of the lattice  $T$ , we denote this  $k_L$ -representation by  $\bar{V}$ . Given an integer  $1 \leq s \leq p$ , we denote by  $\text{ind } \omega_2^s$  the unique 2-dimensional  $k_L$ -representation  $\rho$  of  $\mathcal{G}_{\mathbb{Q}_p}$  such that  $\det \rho = \omega^s$  and  $\rho|_{\mathcal{I}_{\mathbb{Q}_p}} \cong \omega_2^s \oplus \omega_2^{ps}$ , where  $\omega_2$  is the fundamental character of level 2, then  $\text{ind } \omega_2^s$  is absolutely irreducible, and any absolutely irreducible 2-dimensional  $k_L$ -representation of  $\mathcal{G}_{\mathbb{Q}_p}$  is isomorphic to a twist of  $\text{ind } \omega_2^s$ , for some  $1 \leq s \leq p$ .

Recall that a representation  $V$  of  $\mathcal{G}_{\mathbb{Q}_p}$  is crystabelline if it becomes crystalline after restriction to  $\text{Gal}(\bar{\mathbb{Q}}_p/E)$ , where  $E$  is an abelian extension of  $\mathbb{Q}_p$ . Absolutely irreducible  $L$ -linear 2-dimensional crystabelline representations of  $\mathcal{G}_{\mathbb{Q}_p}$  with Hodge-Tate weights  $(0, k-1)$ , ( $k \geq 2$ ) can be parameterized by pairs of smooth characters  $\alpha, \beta : \mathbb{Q}_p^\times \rightarrow L^\times$ , such that  $-(k-1) < \text{val}(\alpha(p)) \leq \text{val}(\beta(p)) < 0$  and  $\text{val}(\alpha(p)) + \text{val}(\beta(p)) = -(k-1)$ ,

see [4, Prop 2.4.5] or [16, §5.5]. We denote by  $V(\alpha, \beta)$  the unique crystabelline representation  $V$ , such that  $D_{\text{cris}}(V) = D(\alpha, \beta)$ , where  $D(\alpha, \beta)$  is the filtered admissible  $L$ -linear  $(\varphi, \mathcal{G}_{\mathbb{Q}_p})$ -module defined in [4, Def 2.4.4].

**Theorem 8.6.** *Fix an integer  $k \geq 2$  and smooth characters  $\theta_1, \theta_2 : \mathbb{Z}_p^\times \rightarrow L^\times$ . Let  $M$  be a  $K$ -stable lattice in  $\tau(\theta_1, \theta_2) \otimes \text{Sym}^{k-2} L^2$ . Suppose that  $\sigma := \text{Sym}^r k_L^2 \otimes \det^a$  with  $0 \leq r \leq p-1$  and  $0 \leq a < p-1$  is a subquotient of  $M \otimes_A k_L$ . Let  $\rho$  be one of the following:*

- (a)  $\rho = (\text{ind } \omega_2^{r+1}) \otimes \omega^a$ ;
- (b) if  $(\omega^{r+1+a} \oplus \omega^a)|_{\mathcal{I}_{\mathbb{Q}_p}}$  is not isomorphic to either  $\bar{\theta}_1 \oplus \bar{\theta}_2 \omega^{k-1}$  or  $\bar{\theta}_2 \oplus \bar{\theta}_1 \omega^{k-1}$  then let  $\rho = \mu_\lambda \omega^{r+1+a} \oplus \mu_{\lambda^{-1}} \omega^a$ , for any  $\lambda \in k_L^\times$ .

Then there exists a finite extension  $L'$  of  $L$  and an absolutely irreducible crystabelline  $L'$ -representation  $V := V(\alpha, \beta)$  such that the following hold:

- (1)  $\bar{V} \cong \rho$ ;
- (2)  $\alpha(p)\beta(p)p^{k-1} = 1$ ;
- (3) the Hodge-Tate weights of  $V$  are 0 and  $k-1$ ;
- (4) either  $(\alpha|_{\mathbb{Z}_p^\times} = \theta_1 \text{ and } \beta|_{\mathbb{Z}_p^\times} = \theta_2)$  or  $(\alpha|_{\mathbb{Z}_p^\times} = \theta_2 \text{ and } \beta|_{\mathbb{Z}_p^\times} = \theta_1)$ .

*Proof.* In case (a) set  $\kappa := \kappa(r, \omega^a)$ . In case (b) if  $(r, \lambda) = (0, \pm 1)$  then set  $\kappa := (\mu_{\pm 1} \omega^a) \circ \det$ ; if  $(r, \lambda) = (p-1, \pm 1)$ , then set  $\kappa := \text{Sp} \otimes (\mu_{\pm 1} \omega^a) \circ \det$ ; otherwise set  $\kappa := \text{Ind}_B^G \mu_{\lambda^{-1}} \omega^a \otimes \mu_\lambda \omega^{a+r}$ . Lemma 8.1 implies that  $\kappa$  can be realized over  $k_L$ . Moreover, it follows from Lemma 8.2 that  $\text{Hom}_K(\sigma, \kappa) \neq 0$ . Theorem 8.5 gives an admissible unitary completion  $E$  of  $\text{Ind}_B^G \chi_1 \otimes \chi_2 | \cdot |^{-1} \otimes \text{Sym}^{k-2}(L')^2$  with  $\chi_1, \chi_2$  and  $E$  satisfying conditions (1)-(5) of Theorem 8.5. We note that  $\kappa$  is not a subquotient of  $\text{Ind}_B^G \psi_1 \otimes \psi_2$ , with  $(\psi_1|_{\mathbb{Z}_p^\times}, \psi_2|_{\mathbb{Z}_p^\times}) = (\bar{\theta}_1, \bar{\theta}_2 \omega^{k-2})$  and  $(\psi_1|_{\mathbb{Z}_p^\times}, \psi_2|_{\mathbb{Z}_p^\times}) = (\bar{\theta}_2, \bar{\theta}_1 \omega^{k-2})$ . In case (a) this is automatic, since  $\kappa$  is supersingular hence not a subquotient of any principal series, and in case (b) this follows from the assumption and Lemma 8.4. In particular, we have  $\text{val}(\chi_1(p)) < 0$  and  $\text{val}(\chi_2(p)) < 0$ . If  $\text{val}(\chi_1(p)) \leq \text{val}(\chi_2(p))$  then set  $\alpha := \chi_1$  and  $\beta := \chi_2$ , otherwise set  $\alpha := \chi_2$  and  $\beta := \chi_1$ , so that  $\text{val}(\alpha(p)) \leq \text{val}(\beta(p))$ .

If  $\chi_1 = \chi_2 | \cdot |$ , then  $\chi_1(p) = \chi_2(p)p^{-1}$ , and so  $\text{val}(\chi_1(p)) < \text{val}(\chi_2(p))$ . If  $\chi_1 = \chi_2 | \cdot |^{-1}$  then the representation  $\text{Ind}_B^G \chi_1 \otimes \chi_2 | \cdot |^{-1}$  has  $\chi_1 \circ \det$  as a subobject. Hence,  $\text{Sym}^{k-2}(L')^2 \otimes \chi_1 \circ \det$  admits a  $G$ -invariant lattice, which implies  $k = 2$ . In particular,  $\theta_1 = \theta_2$ ,  $r = 0$  and  $\omega^a = \bar{\theta}_1$ . The assumption in

(b), implies that we are in case (a), so that  $\rho \cong (\text{ind } \omega_2) \otimes \omega^a$ . Since  $k = 2$  and  $p > 2$  it follows from [2] that if  $V = V(\chi_1 | \cdot |^{-1}, \chi_1)$  then  $\overline{V} \cong \rho$ , see the example below. Assume that  $\chi_1 \neq \chi_2 | \cdot |^{\pm 1}$  then we have an isomorphism

$$\text{Ind}_B^G \chi_1 \otimes \chi_2 | \cdot |^{-1} \cong \text{Ind}_B^G \chi_2 \otimes \chi_1 | \cdot |^{-1}.$$

So without loss of generality we may assume that  $E$  is a unitary admissible completion of  $\pi \otimes \text{Sym}^{k-2} L^2$ ,  $\pi := \text{Ind}_B^G \alpha \otimes \beta | \cdot |^{-1}$  with

- (i)  $\alpha \neq \beta$ ;
- (ii)  $\alpha(p)\beta(p)p^{k-1} = 1$ ;
- (iii)  $-(k-1) < \text{val}(\alpha(p)) \leq \text{val}(\beta(p)) < 0$ .

In this situation, Berger-Breuil have shown that the completion of  $\pi \otimes \text{Sym}^{k-2} L^2$  with respect to any finitely generated  $A[G]$ -lattice is topologically irreducible, [4, Cor 5.3.2, 5.3.4]. This implies that the completion  $E$  is topologically irreducible and is isomorphic as a unitary  $L$ -Banach space representation of  $G$  to the representation  $B(V)$ , with  $V := V(\alpha, \beta)$ , defined in [4, Def 4.2.4]. In [5] Berger has shown that there are two possibilities:

- (A)  $E^0 \otimes_A k \cong \kappa(s, b)$ , with  $0 \leq s \leq p-1$  and  $0 \leq b < p-1$ , in which case  $\overline{V} \cong (\text{ind } \omega_2^{s+1}) \otimes \omega^b$ ;
- (B)  $(E^0 \otimes_A k)^{ss} \cong (\text{Ind}_B^G \psi_1 \otimes \psi_2 \omega^{-1})^{ss} \oplus (\text{Ind}_B^G \psi_2 \otimes \psi_1 \omega^{-1})^{ss}$ , in which case  $\overline{V} \cong \psi_1 \oplus \psi_2$ .

Since, we know that  $\text{Hom}_G(\kappa, E^0 \otimes_A k) \neq 0$ , the result of Berger together with [1, Thm 33, 34], [9, Cor 4.1.4] implies that  $\overline{V} \cong \rho$ .  $\square$

**Example.** Assume that  $\theta_1 = \theta_2 = \mathbf{1}$ , so that  $\tau(\theta_1, \theta_2)$  is the trivial representation of  $K$ . Fix an integer  $k \geq 2$ , and choose  $\alpha_p, \beta_p \in \mathfrak{M}$ , such that  $\alpha_p \beta_p = p^{k-1}$ , set  $a_p := \alpha_p + \beta_p$ . We may assume that  $\text{val}(\alpha_p) \geq \text{val}(\beta_p)$ , define unramified characters  $\alpha, \beta : \mathbb{Q}_p^\times \rightarrow L^\times$ , by  $\alpha(p) := \alpha_p^{-1}$  and  $\beta(p) := \beta_p^{-1}$ . The representation  $V := V(\alpha, \beta)$  is crystalline with Hodge-Tate weights  $(0, k-1)$ , and is isomorphic to the representation denoted by  $V_{k, a_p}$  in [5], [2]. In [2] and [5] the reduction  $\overline{V}$  is computed when  $2 \leq k \leq 2p+1$ , (see also [3], the case  $k = 2p+1$  is an unpublished result of Breuil). We will illustrate the Theorem in this case. Let  $M$  be a  $K$ -stable lattice in  $\text{Sym}^{k-2} L^2$ , with  $2 \leq k \leq 2p+1$ . Let  $\sigma := \text{Sym}^r k_L^2 \otimes \det^a$  be an irreducible subquotient of  $M \otimes_A k$  and let  $\rho$  be as in Theorem 8.6. We will show that the assertion of Theorem 8.6 matches

the computations of [2], [5], that is there exists  $a_p \in \mathfrak{M}$  such that  $\overline{V}_{k,a_p} \cong \rho$ . We note that the assumption in Theorem 8.6 (b) implies that we exclude the representations  $\rho$ , such that  $\rho|_{\mathcal{I}_{\mathbb{Q}_p}} \cong \mathbf{1} \oplus \omega^{k-1}$ .

If  $2 \leq k \leq p+1$  then  $(M \otimes_A k_L)^{ss} \cong \text{Sym}^{k-2} k_L^2$ , and hence  $\rho = \text{ind } \omega_2^{k-1}$ . Now it follows from [2, Cor 4.1.3, Prop. 4.1.4] that  $\overline{V} \cong \rho$ .

If  $k = p+2$  then (13) below gives  $(M \otimes_A k_L)^{ss} \cong \text{Sym}^1 k_L^2 \oplus \text{Sym}^{p-2} k_L^2 \otimes \det$ , so  $\rho$  is either  $\text{ind } \omega_2^2$ ,  $(\text{ind } \omega_2^{p-1}) \otimes \omega \cong \text{ind } \omega^2$ , [9, Lem. 4.2.2] or  $\mu_\lambda \omega^p \oplus \mu_{\lambda^{-1}} \omega = \mu_\lambda \omega \oplus \mu_{\lambda^{-1}} \omega$ , for  $\lambda \in k_L^\times$ . If  $0 < \text{val}(a_p) < 1$  then  $\overline{V} \cong \text{ind } \omega_2^2$ , [5, Thm 3.2.1]. If  $\text{val}(a_p) = 1$  then  $\overline{V} \cong \omega \mu_\lambda \oplus \omega \mu_{\lambda^{-1}}$ , where  $\lambda$  is a root of a polynomial  $X^2 - \overline{a_p/p}X + 1$ . Now  $\alpha_p \beta_p = p^{p+1}$ ,  $\text{val}(\alpha_p) \geq \text{val}(\beta_p)$ ,  $\text{val}(\alpha_p + \beta_p) = 1$  implies that  $\text{val}(\beta_p) = 1$  and  $\text{val}(\alpha_p) > 1$ . So  $a_p/p \equiv \beta_p/p \pmod{\mathfrak{M}}$ . Choose any  $u \in k_L^\times$ , and let  $[u]$  denote the Teichmüller lift. Replace  $\alpha_p$  with  $\alpha_p[u]^{-1}$  and  $\beta_p$  with  $\beta_p[u]$ . Then  $\overline{V}$  is isomorphic to  $\omega \mu_{\lambda_u} \oplus \omega \mu_{\lambda_u^{-1}}$ , where  $\lambda_u$  is any root of the polynomial  $X^2 + u \overline{a_p/p}X + 1$ . Given  $\xi \in k_L^\times$ , such that  $\xi^2 \neq -1$ , let  $u = -(\xi + \xi^{-1})\overline{p/a_p}$ , then  $\lambda_u = \xi^{\pm 1}$ . If  $\text{val}(a_p) > 1$  then  $\overline{V} \cong \mu_{\sqrt{-1}} \omega \oplus \mu_{-\sqrt{-1}} \omega$ , [2, Cor. 4.1.3, Prop. 4.1.4].

If  $p+3 \leq k \leq 2p$ , then set  $r_0 := (k-2) - (p-1)$ ,  $r_1 := (k-2) - (p+1)$ . We have  $2 \leq r_0 \leq p-1$  and  $0 \leq r_1 \leq p-3$ . Then

$$(M \otimes_A k_L)^{ss} \cong (\text{Sym}^{r_1} k_L^2 \otimes \det) \oplus \text{Sym}^{r_0} k_L^2 \oplus (\text{Sym}^{p-1-r_0} k_L^2 \otimes \det^{r_0})$$

see (15). Then  $\rho$  is  $\text{ind } \omega_2^{r_0+1} \cong (\text{ind } \omega_2^{p-r_0}) \otimes \omega^{r_0}$ , or  $(\text{ind } \omega_2^{r_1+1}) \otimes \omega$ , or  $\mu_\lambda \omega^{r_1+2} \oplus \mu_{\lambda^{-1}} \omega \cong \mu_\lambda \omega^{k-2} \oplus \mu_{\lambda^{-1}} \omega$ , for some  $\lambda \in k_L^\times$ . If  $0 < \text{val}(a_p) < 1$  then  $\overline{V} \cong \text{ind } \omega^{r_0+1}$ , if  $\text{val}(a_p) > 1$  then  $\overline{V} \cong (\text{ind } \omega_2^{r_1+1}) \otimes \omega$ , and if  $\text{val}(a_p) = 1$  then  $\overline{V} \cong \mu_\lambda \omega^{k-2} \oplus \mu_{\lambda^{-1}} \omega$ , where  $\lambda = (k-1)\overline{a_p/p}$ , [5, Thm 3.2.1]. Again we see that every  $\rho$  is isomorphic to some  $\overline{V}$ .

If  $k = 2p+1$  then (15) gives

$$(M \otimes_A k_L)^{ss} \cong (\text{Sym}^{p-2} k_L^2 \otimes \det) \oplus \text{Sym}^1 k_L^2 \oplus (\text{Sym}^{p-2} k_L^2 \otimes \det).$$

So the possibilities for  $\rho$  are the same as in the case  $k = p+2$ , so that  $\rho = \text{ind } \omega_2^2$  or  $\rho = \mu_\lambda \omega \oplus \mu_{\lambda^{-1}} \omega$ , for  $\lambda \in k_L^\times$ . If  $\text{val}(a_p^2 + p) < 3/2$  then  $\overline{V} \cong \text{ind } \omega_2^2$ , if  $\text{val}(a_p^2 + p) \geq 3/2$  then  $\overline{V} \cong \omega \mu_\lambda \oplus \omega \mu_{\lambda^{-1}}$ , where  $\lambda$  is any root of  $X^2 + \frac{a_p^2 + p}{2pa_p}X + 1$ , [5, Thm 3.2.1]. We leave it as an exercise to an interested reader to check that given  $\xi \in k_L^\times$  there exists  $\alpha_p, \beta_p$ , such that  $\alpha_p \beta_p = p^{2p}$ ,  $\text{val}((\alpha_p + \beta_p)^2 + p) \geq 3/2$  and  $\xi + \xi^{-1} = -\frac{(\alpha_p + \beta_p)^2 + p}{2p(\alpha_p + \beta_p)}$ .

**Lemma 8.7.** *Let  $\theta_1, \theta_2 : \mathbb{Z}_p^\times \rightarrow L^\times$  be smooth characters and  $k \geq 2$  an integer. Let  $M$  be a  $K$ -stable lattice in  $\tau(\theta_1, \theta_2) \otimes \text{Sym}^{k-2} L^2$ . We make the following assumptions:*

- (a) if  $\theta_1 = \theta_2$  then assume  $k \geq p^2 + 1$ ;
- (b) if  $\theta_1 \neq \theta_2$  and  $\theta_1 \theta_2^{-1}$  is trivial on  $1 + p\mathbb{Z}_p$  then assume  $k \geq p$ .

Then every irreducible  $k_L$ -representation  $\sigma$  of  $K$  with the central character  $\bar{\theta}_1 \bar{\theta}_2 \omega^{k-2}$  is a subquotient of  $M \otimes_A k$ .

*Proof.* This is shown in the appendix.  $\square$

**Corollary 8.8.** Fix smooth characters  $\theta_1, \theta_2 : \mathbb{Z}_p^\times \rightarrow L^\times$ , and an integer  $k \geq 2$ , such that

- (a) if  $\theta_1 = \theta_2$  then assume  $k \geq p^2 + 1$ ;
- (b) if  $\theta_1 \neq \theta_2$  and  $\theta_1 \theta_2^{-1}$  is trivial on  $1 + p\mathbb{Z}_p$  then assume  $k \geq p$ .

Let  $\rho$  be a semisimple 2-dimensional  $k_L$ -representation of  $\mathcal{G}_{\mathbb{Q}_p}$ , such that

- (c)  $\det \rho|_{\mathcal{I}_{\mathbb{Q}_p}} = \bar{\theta}_1 \bar{\theta}_2 \omega^{k-1}$ ;
- (d) if  $\rho$  is irreducible, then it is absolutely irreducible;
- (e)  $\rho|_{\mathcal{I}_{\mathbb{Q}_p}} \not\cong \bar{\theta}_1 \oplus \bar{\theta}_2 \omega^{k-1}$  and  $\rho|_{\mathcal{I}_{\mathbb{Q}_p}} \not\cong \bar{\theta}_2 \oplus \bar{\theta}_1 \omega^{k-1}$ .

Then there exists a finite extension  $L'$  of  $L$  and an absolutely irreducible 2-dimensional crystabelline  $L'$ -representation  $V := V(\alpha, \beta)$  of  $\mathcal{G}_{\mathbb{Q}_p}$ , such that

- (i)  $\bar{V} \cong \rho$ ;
- (ii) Hodge-Tate weights of  $V$  are  $(0, k-1)$ ;
- (iii) either  $(\alpha|_{\mathbb{Z}_p^\times} = \theta_1 \text{ and } \beta|_{\mathbb{Z}_p^\times} = \theta_2)$  or  $(\alpha|_{\mathbb{Z}_p^\times} = \theta_2 \text{ and } \beta|_{\mathbb{Z}_p^\times} = \theta_1)$ .

*Proof.* After twisting by a character, we can get  $\rho$  to be as in Theorem 8.6. The assertion follows from Theorem 8.6 and Lemma 8.7.  $\square$

## A Semi-simplification

We prove Lemma 8.7. To simplify the notation we set  $n := k - 2$  we keep the assumption  $p > 2$  and notations of the previous section. Let  $M$  be a  $K$ -invariant lattice in  $\tau(\theta_1, \theta_2) \otimes \text{Sym}^n L^2$ . Since  $\tau(\theta_1, \theta_2) \otimes \text{Sym}^n L^2$  is a finite dimensional  $L$ -vector space,  $(M \otimes_A k_L)^{ss}$  does not depend on the choice of  $M$ , see the proof of [37, Thm 32]. Since  $\theta_1$  and  $\theta_2$  are smooth characters,  $\theta_1(g)$  and  $\theta_2(g)$  are roots of unity for all  $g \in \mathbb{Z}_p^\times$ . Hence,  $\theta_1$  and  $\theta_2$  are  $A$ -valued. If  $\delta : \mathbb{Z}_p^\times \rightarrow L^\times$  is a smooth character then Lemma 8.7 holds for  $\theta_1, \theta_2, k$  if and only if it holds for  $\theta_1\delta, \theta_2\delta, k$ . In particular, if  $c = 0$  we may assume that  $\theta_1 = \theta_2 = \mathbf{1}$ , so that  $\tau(\theta_1, \theta_2)$  is the trivial representation, and take  $M := \text{Sym}^n A^2$ , so that  $M \otimes_A k_L \cong \text{Sym}^n k_L^2$ . If  $c > 1$  let  $M_1$  be the space of functions  $f : K \rightarrow A$ , such that  $f(hg) = (\theta_1 \otimes \theta_2)(h)f(g)$ , for all  $g \in K, h \in J_c$ . Then  $M_1$  is a  $K$ -invariant lattice in  $\tau(\theta_1, \theta_2)$ , so  $M := M_1 \otimes_A \text{Sym}^n A$  is a  $K$ -invariant lattice in  $\tau(\theta_1, \theta_2) \otimes \text{Sym}^n L^2$ , and  $M \otimes_A k_L \cong (\text{Ind}_{J_c}^K \bar{\theta}_1 \otimes \bar{\theta}_2) \otimes \text{Sym}^n k_L^2$ . Since  $k_L$  contains  $\mathbb{F}_p$  every irreducible representation is absolutely irreducible. Hence, as far as semi-simplification is concerned working over  $k_L$  is the same as working over an algebraically closed field, see [37, §14.6].

We first look at the case  $c = 0$  and so  $M \otimes_A k_L \cong \text{Sym}^n k_L^2$ . Since  $K_1$  acts trivially on  $\text{Sym}^n k_L^2$ , it is enough to compute the semi-simplification of  $\text{Sym}^n k_L^2$  as a representation of  $\text{GL}_2(\mathbb{F}_p)$ . Recall that semi-simplification is determined by the Brauer character, which is a  $\overline{\mathbb{Q}}_p$ -valued function on  $p$ -regular conjugacy classes of  $\text{GL}_2(\mathbb{F}_p)$ , [37, §18.2]. We have [21, §1]:

$$\chi_n\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = (n+1)[\lambda]^n, \quad \chi_n\left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\right) = \frac{[\lambda]^{n+1} - [\mu]^{n+1}}{[\lambda] - [\mu]},$$

where  $\lambda, \mu \in \mathbb{F}_p^\times$  and  $\lambda \neq \mu$ . Moreover, choose an embedding  $\iota : \mathbb{F}_{p^2}^\times \rightarrow \text{GL}_2(\mathbb{F}_p)$ , suppose that  $z \in \mathbb{F}_p^2 \setminus \mathbb{F}_p$  then  $\chi_n(\iota(z))$  does not depend on  $\iota$  and we have:

$$\chi_n(z) = [z]^n \frac{[z]^{(p-1)(n+1)} - 1}{[z]^{p-1} - 1}.$$

**Lemma A.1.** *Let  $n \geq p + 1$  be an integer then  $\chi_n = \chi_{n-p-1} \det + \chi_r + \chi_{p-r-1} \det^r$ , where  $0 \leq r < p - 1$  and  $n \equiv r \pmod{p-1}$ .*

*Proof.* Let  $\psi := \chi_n - \chi_{n-p-1} \det$  then a calculation using the formulae above gives:

$$\psi\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = (p+1)[\lambda]^r, \quad \psi\left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\right) = [\lambda]^r + [\mu]^r, \quad \psi(z) = 0.$$

Let  $B(\mathbb{F}_p)$  be the group of upper triangular matrices in  $\mathrm{GL}_2(\mathbb{F}_p)$  and let  $\chi : B(\mathbb{F}_p) \rightarrow \overline{\mathbb{F}}_p^\times$  be the character, given by  $\chi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = a^r$ . It follows from [21, §1] that  $\psi$  is the Brauer character of  $\mathrm{Ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)} \chi$ . Since

$$(\mathrm{Ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)} \chi)^{ss} \cong \mathrm{Sym}^r k_L^2 \oplus \mathrm{Sym}^{p-r-1} k_L^2 \otimes \det^r,$$

see eg. [30, Lem 3.1.7, 4.1.3], we obtain the result.  $\square$

If  $0 \leq n \leq p-1$  then  $\mathrm{Sym}^n \overline{\mathbb{F}}_p^2$  is irreducible, if  $n = p$  then

$$(\mathrm{Sym}^p k_L^2)^{ss} \cong \mathrm{Sym}^1 k_L^2 \oplus \mathrm{Sym}^{p-2} k_L^2 \otimes \det, \quad (13)$$

see [10, Lem.5.1.3]. Using Lemma A.1 and (13) we may compute the semi-simplification. Let  $m$  be the largest integer such that  $n \geq (p+1)m$ , for  $0 \leq i \leq m$  let  $0 \leq r_i < p-1$  be the unique integer such that  $n - (p+1)i \equiv r_i \pmod{p-1}$ . If  $n - (p+1)m = p$  then

$$(\mathrm{Sym}^n k_L^2)^{ss} \cong \bigoplus_{i=0}^m (\mathrm{Sym}^{r_i} k_L^2 \otimes \det^i \oplus \mathrm{Sym}^{p-r_i-1} k_L^2 \otimes \det^{r_i+i}), \quad (14)$$

otherwise,  $(\mathrm{Sym}^n k_L^2)^{ss}$  is isomorphic to

$$\mathrm{Sym}^{r_m} k_L^2 \otimes \det^m \oplus \bigoplus_{i=0}^{m-1} (\mathrm{Sym}^{r_i} k_L^2 \otimes \det^i \oplus \mathrm{Sym}^{p-r_i-1} k_L^2 \otimes \det^{r_i+i}). \quad (15)$$

**Lemma A.2.** *Assume  $p > 2$  and let  $n \geq p^2 - 1$  be an integer, let  $\sigma$  be an irreducible  $k_L$ -representation of  $K$ , with central character  $\omega^n$ . Then  $\sigma$  occurs as a subquotient of  $\mathrm{Sym}^n k_L^2$ .*

*Proof.* We note that the central character of  $\mathrm{Sym}^n k_L^2$  is  $\omega^n$ . Hence, every irreducible subquotient will also have a central character  $\omega^n$ . We have  $\sigma \cong \mathrm{Sym}^r k_L^2 \otimes \det^a$  with  $0 \leq r \leq p-1$  and  $0 \leq a < p-1$ . The central character of  $\sigma$  is equal to  $\omega^{r+2a}$ . The equality  $\omega^{r+2a} = \omega^n$  implies that 2 divides  $n-r$ . Let  $r_j$  be as above and note that  $r_j \equiv r_0 - 2j \pmod{p-1}$ . Since  $p+1$  is even we get that 2 divides  $r-r_0$ . Let  $0 \leq j < (p-1)/2$  be the unique integer such that  $r_0 - 2j \equiv r \pmod{p-1}$ . Then  $r = r_j = r_{j+(p-1)/2}$ . The congruence,  $2a \equiv n-r \equiv r_0-r \pmod{p-1}$  implies that either  $a = j$  or  $a = j + (p-1)/2$ . Hence, either  $\sigma \cong \mathrm{Sym}^{r_j} k_L^2 \otimes \det^j$  or  $\sigma \cong \mathrm{Sym}^{r_{j+(p-1)/2}} k_L^2 \otimes \det^{j+(p-1)/2}$ . It follows from (14) and (15) that  $\sigma$  is an irreducible subquotient of  $\mathrm{Sym}^n k_L^2$ . Note that the assumption  $n \geq p^2 - 1$  implies that  $m \geq p-1$ .  $\square$



We look at the case  $c \geq 1$ , so that  $M \otimes_A k_L \cong (\text{Ind}_{J_c}^K \bar{\theta}_1 \otimes \bar{\theta}_2) \otimes \text{Sym}^n k_L^2$ .

**Lemma A.3.** *If  $\theta_1 \theta_2^{-1}$  is trivial on  $1 + p\mathbb{Z}_p$  then assume  $n \geq p - 2$ . Then every irreducible  $k_L$ -representation  $\sigma$  of  $K$  with the central character  $\bar{\theta}_1 \bar{\theta}_2 \omega^{k-2}$  is a subquotient of  $M \otimes_A k$ .*

*Proof.* We note that the central character of  $M \otimes_A k$  is  $\bar{\theta}_1 \bar{\theta}_2 \omega^n$ , hence every irreducible subquotient of  $M \otimes_A k$  will have the central character  $\bar{\theta}_1 \bar{\theta}_2 \omega^n$ . Set  $Z_K := Z \cap K$  then every irreducible  $\sigma$  with the central character  $\bar{\theta}_1 \bar{\theta}_2 \omega^{k-2}$  will be a subquotient of

$$\text{Ind}_{Z_K I_1}^K \bar{\theta}_1 \bar{\theta}_2 \omega^n \cong \bigoplus_{i=0}^{p-2} \text{Ind}_I^K \omega^{n-i} \bar{\theta}_1 \otimes \omega^i \bar{\theta}_2. \quad (16)$$

Now

$$M \otimes_A k \cong \text{Ind}_{J_c}^K ((\bar{\theta}_1 \otimes \bar{\theta}_2) \otimes \text{Sym}^n k_L^2). \quad (17)$$

Since  $((\text{Sym}^n k_L^2)|_{J_c})^{ss} \cong \bigoplus_{i=0}^n (\omega^{n-i} \otimes \omega^i)$ , if  $c = 1$  and  $n \geq p - 2$  then  $J_c = I$  and (16) and (17) give the claim. Assume that  $c > 1$  then  $(M \otimes_A k_L)^{ss}$  will contain

$$(\text{Ind}_{J_2}^K \bar{\theta}_1 \omega^n \otimes \bar{\theta}_2)^{ss} \cong (\text{Ind}_I^K ((\bar{\theta}_1 \omega^n \otimes \bar{\theta}_2) \otimes \text{Ind}_{J_2}^I \mathbf{1}))^{ss}.$$

Now  $(\text{Ind}_{J_2}^I \mathbf{1})^{ss} \cong \bigoplus_{i=0}^{p-1} (\omega^{-i} \otimes \omega^i)$ . The claim follows from (16).  $\square$

## References

- [1] L. BARTHEL AND R. LIVNÉ, 'Irreducible modular representations of  $\text{GL}_2$  of a local field', *Duke Math. J.* 75, no 2, 1994.
- [2] L. BERGER, H. LI AND H. ZHU 'Construction of some families of 2-dimensional crystalline representations', *Mathematische Annalen* 329 (2004), no. 2, 365–377.
- [3] L. BERGER AND C. BREUIL, 'Sur la réduction des représentations cristallines de dimension 2 en poids moyens', preprint.
- [4] L. BERGER AND C. BREUIL, 'Sur quelques représentations potentiellement cristallines de  $\text{GL}_2(\mathbb{Q}_p)$ ', to appear in *Astérisque*.
- [5] L. BERGER, 'Représentations modulaires de  $\text{GL}_2(\mathbb{Q}_p)$  et représentations galoisiennes de dimension 2', to appear in *Astérisque*.

- [6] J.-N. BERNSTEIN, (rédigé par P. DELIGNE), 'Le 'centre' de Bernstein', *Représentations des groupes réductifs sur un corps local*, Herman, Paris 1984, 1-32.
- [7] N. BOURBAKI, *Commutative Algebra*, Paris, Hermann 1972.
- [8] C. BREUIL AND A. MÉZARD, 'Multiplicités modulaires et représentations de  $GL_2(\mathbb{Z}_p)$  et de  $Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$  en  $l = p$ ', *Duke Math. J.* 115, 2002, 205-310.
- [9] C. BREUIL, 'Sur quelques représentations modulaires et  $p$ -adiques de  $GL_2(\mathbb{Q}_p)$ . I', *Compositio* 138, 165-188, 2003.
- [10] C. BREUIL, 'Sur quelques représentations modulaires et  $p$ -adiques de  $GL_2(\mathbb{Q}_p)$ . II', *J. Inst. Math. Jussieu* 2, 1-36, 2003.
- [11] C. BREUIL AND P. SCHNEIDER, 'First steps towards  $p$ -adic Langlands functoriality', *J. Reine Angew. Math.* 610, 2007, 149-180.
- [12] C. BREUIL AND V. PAŠKŪNAS, 'Towards a modulo  $p$  Langlands correspondence for  $GL_2$ ', preprint 2007.
- [13] A. BRUMER, 'Pseudocompact algebras, profinite groups and class formations', *J. of Algebra*, 4, 442-470 (1966).
- [14] D. BUMP, *Automorphic forms and representations*, Cambridge Studies in Advanced Math. 55, CUP, 1998.
- [15] C. J. BUSHNELL AND P. C. KUTZKO 'Smooth representations of reductive  $p$ -adic groups: structure theory via types', *Proc. London Math. Soc.* (3) 77 (1988) 582-634.
- [16] P. COLMEZ, 'Série principale unitaire pour  $GL_2(\mathbb{Q}_p)$  et représentations triangulines de dimension 2', preprint 2004.
- [17] P. COLMEZ, 'Représentations de  $GL_2(\mathbb{Q}_p)$  et  $(\varphi, \Gamma)$ -modules', preprint (2007).
- [18] C.W. CURTIS AND I. REINER, *Methods of representation theory. Volume I*, John Wiley and Sons, 1981.
- [19] M. DEMAZURE AND P. GABRIEL, *Groupes algébriques*, Tome I, Masson and Cie, Paris, 1970.
- [20] M. DEMAZURE AND A. GROTHENDIECK, *Schémas en Groupes I*. Lect. Notes Math 151, Springer 1970.

- [21] F. DIAMOND 'A correspondence between representations of local Galois groups and Lie-type groups', *Proceedings of the LMS Durham Symposium on L-functions and Galois Representations*, 2004.
- [22] M. EMERTON 'Locally analytic vectors in representations of locally  $p$ -adic analytic groups', to appear in *Memoirs of the AMS*.
- [23] M. EMERTON 'Locally analytic representation theory of  $p$ -adic reductive groups: A summary of some recent developments *L-functions and Galois representations* (D.Burns, K.Buzzard and J.Nekovář, eds.), London Mathematical Society Lecture Note Series, vol.320, 2007, 407-437.
- [24] P. GABRIEL, 'Des Catégories abéliennes', *Bull. Math. Soc. France*, 90, 1962, 323-448.
- [25] G. HENNIART, 'Sur l'unicité des types pour  $GL_2$ ', appendix to [8].
- [26] S. LANG, *Algebra*, Revised third edition, Springer 2002.
- [27] M. LAZARD, 'Groupes analytiques  $p$ -adiques', *Publ. Math. IHES* 26 (1965).
- [28] S. MAC LANE, *Homology*, Grundlehren der mathematischen Wissenschaften 114, Springer 1975.
- [29] J. NEUKIRCH, A. SCHMIDT, K. WINGBERG, *Cohomology of number fields*, Springer 2000.
- [30] V. PAŠKŪNAS, 'Coefficient systems and supersingular representations of  $GL_2(F)$ ', *Mémoires de la SMF*, 99, 2004.
- [31] V. PAŠKŪNAS, 'On the restriction of representations of  $GL_2(F)$  to a Borel subgroup', *Compositio* 143, 1533-1544, (2007).
- [32] V. PAŠKŪNAS, 'On some crystalline representations of  $GL_2(\mathbb{Q}_p)$ ', preprint 2008.
- [33] D. PRASAD, 'Locally algebraic representations of  $p$ -adic groups', appendix to [35].
- [34] P. SCHNEIDER, *Nonarchimedean Functional Analysis*, Springer (2001).
- [35] P. SCHNEIDER AND J. TEITELBAUM, ' $U(\mathfrak{g})$ -finite locally analytic representations', *Representation Theory* 5, 111-128 (2001).

- [36] P. SCHNEIDER AND J. TEITELBAUM 'Banach space representations and Iwasawa theory', *Israel J. Math.* 127, 359-380 (2002).
- [37] J.-P. SERRE *Linear representation of finite groups*, Springer (1977).
- [38] O. VENJAKOB 'Characteristic Elements in Noncommutative Iwasawa Theory', *J. reine angew. Math.* 583 (2005).
- [39] M.-F. VIGNÉRAS, *Répresentations  $l$ -modulaires d'un groupe réductif  $p$ -adique avec  $l \neq p$* , Progress in Math., Birkhäuser, Boston, 1996.
- [40] M.-F. VIGNÉRAS 'Admissibilité des représentations  $p$ -adiques et lemme de Nakayama', preprint 2007.
- [41] M.-F. VIGNÉRAS, 'A criterion for integral structures and coefficient systems on the tree of  $PGL(2, F)$ ', *Pure and Applied Mathematics Quarterly*, Volume 4, Number 4, 2008 (Jean-Pierre Serre special issue, part I).